Capacitary problems in fibered structures.

Michel Bellieud

LMGC UMR 5508, University of Montpellier, France

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Related problems.

Perforated domain with Dirichlet conditions: Marchenko, Khruslov 78, Cioranescu, Murat 82 (cloud of ice), Allaire 91 (Fluid flow past fixed solid obstacles), Dal Maso (94...), Garroni, Casado Diaz, Müller...

Composites comprising traces of materials with extreme physical properties: Khruslov 81, Caillerie Dinari 87, B. Bouchitté 98 (diffusion equations), Briane (Stokes) 03, B. Gruais 05, B. 12 (linear elasticity), Bouchitté Felback 06 (Maxwell equations)...

Common point: emergence of a concentration of energy arround the strong components (or holes), in terms of a density of capacity involving the strong components.

This capacity depends on
- the type of the equations,
- the shape the strong components (or holes).

"Classical p-capacity": S bounded domain of $\mathbb{R}^n$, V open set, $\overline{S} \subset V$, $1 < p < +\infty$,

$$\mathrm{cap}(a; S, V) := \inf \left\{ \int_V |\nabla \psi|^p dy, \ \psi \in W_0^{1,p}(V). \quad \psi = a \text{ in } S \right\}.$$

"Elastic capacity".

B. Arma 12 in the spirit of Villaggio Arma 86. $\boldsymbol{a}, \boldsymbol{\alpha} \in \mathbb{R}^3$, $\boldsymbol{y}_S := ⨍_S \boldsymbol{s} ds$,

$$\mathrm{cap}^f(\boldsymbol{a}, \boldsymbol{\alpha}; S, V) := \inf \left\{ \int_V f(\boldsymbol{e}_y(\boldsymbol{\psi}))dy, \begin{array}{l} \boldsymbol{\psi} \in W_0^{1,p}(V;\mathbb{R}^3), \\ \boldsymbol{\psi}(y) = \boldsymbol{a} + \frac{2}{\mathrm{diam} S}\boldsymbol{\alpha} \wedge (\boldsymbol{y} - \boldsymbol{y}_S) \text{ in } S \end{array} \right\}.$$

$$\boldsymbol{e}_y(\boldsymbol{\psi}) := \sum_{\alpha,\beta=1}^{2} \frac{1}{2}\left(\frac{\partial \psi_\alpha}{\partial y_\beta} + \frac{\partial \psi_\beta}{\partial y_\alpha} \right) \boldsymbol{e}_\alpha \otimes \boldsymbol{e}_\beta + \sum_{i=1}^{2} \frac{\partial \psi_3}{\partial y_i} \boldsymbol{e}_i \odot \boldsymbol{e}_3$$

-$\boldsymbol{\psi}$ **not constant in** S**:** $\boldsymbol{\psi}_{\lfloor S} \in \ker \boldsymbol{e}_y$**.**
-Infimum achieved if V is bounded.
- **Fibered structures:** $n = 2$**.**
$\mathrm{cap}^f(\boldsymbol{a}, \boldsymbol{\alpha}; S, V)$: strain energy associated to the **three-dimensional** infinitesimal rigid displacement of reduction elements $\{\boldsymbol{a}, \frac{2}{\mathrm{diam} S}\boldsymbol{\alpha}\}$ at $\boldsymbol{y}_S$ of a **two-dimensional** rigid body S immersed in a two-dimensional space V.

Appropriate notion to describe, in a fibered composite, the concentrations of strain energy generated around the fibers by the three-dimensional infinitesimal rigid displacements of their sections with respect to the embedding matrix.

Basic properties of cap^f.

Assumption: f strictly convex, $\quad c|\boldsymbol{M}|^p \leq f(\boldsymbol{M}) \leq C|\boldsymbol{M}|^p \quad \forall \boldsymbol{M} \in \mathbb{S}^3$.

-Minimizing sequence in $\mathcal{D}(V;\mathbb{R}^3)$.
- $\mathrm{cap}^f(\boldsymbol{a},\boldsymbol{\alpha};S,V) \leq C(S,V)(|\boldsymbol{a}|^p + |\boldsymbol{\alpha}|^p)$,
-$\mathrm{cap}^f(\boldsymbol{a},\boldsymbol{\alpha};S,V)$ convex, continuous w.r.t. $(\boldsymbol{a},\boldsymbol{\alpha})$.
-cap^f decreasing w.r.t. V, increasing w.r.t. S, continuous w.r.t. monotonous sequences.

$$\overline{S} \subset V_1 \subset V_2 \Rightarrow \mathrm{cap}^f(\boldsymbol{a},\boldsymbol{\alpha};S,V_1) \geq \mathrm{cap}^f(\boldsymbol{a},\boldsymbol{\alpha};S,V_2),$$

$$\overline{S}_1 \subset \overline{S}_2 \subset V,\ \boldsymbol{y}_{S_1} = \boldsymbol{y}_{S_2} = 0 \Rightarrow \mathrm{cap}^f\left(\boldsymbol{a}, \frac{\mathrm{diam} S_1}{\mathrm{diam} S_2}\boldsymbol{\alpha}; S_1, V\right) \leq \mathrm{cap}^f(\boldsymbol{a},\boldsymbol{\alpha};S_2,V),$$

$$\bigcup_{n\in\mathbb{N}^*} \uparrow S'_n = S \Rightarrow \lim_{n\to+\infty} \mathrm{cap}^f\left(\boldsymbol{a}, \frac{\mathrm{diam} S'_n}{\mathrm{diam} S}\boldsymbol{\alpha}; S'_n, V\right) = \mathrm{cap}^f(\boldsymbol{a},\boldsymbol{\alpha};S,V),$$

$$\bigcup_{n\in\mathbb{N}^*} \uparrow V_n = V, \bigcap_{n\in\mathbb{N}^*} \downarrow S_n = S$$

$$\Rightarrow \lim_{n\to+\infty} \mathrm{cap}^f\left(\boldsymbol{a}, \frac{\mathrm{diam} S_n}{\mathrm{diam} S}\boldsymbol{\alpha}; S_n, V_n\right) = \mathrm{cap}^f(\boldsymbol{a},\boldsymbol{\alpha};S,V).$$

-f p-positively homogeneous $\Rightarrow \mathrm{cap}^f(\boldsymbol{a},\boldsymbol{\alpha};S,V) = \lambda^{2-p}\mathrm{cap}^f\left(\boldsymbol{a},\boldsymbol{\alpha};\frac{1}{\lambda}S,\frac{1}{\lambda}V\right)$.

Continuity and lower semicontinuity in L^p. Interchange of $\int$ and inf.

$(\boldsymbol{a},\boldsymbol{\alpha}) \to \int_\Omega \mathrm{cap}^f(\boldsymbol{a}(x),\boldsymbol{\alpha}(x);S,V)dx$ strongly continuous,
weakly lower-semicontinuous in $(L^p(\Omega;\mathbb{R}^3))^2$

If $(\boldsymbol{a},\boldsymbol{\alpha}) \in (L^p(\Omega;\mathbb{R}^3))^2$:

$$\int_\Omega \mathrm{cap}^f(\boldsymbol{a}(x),\boldsymbol{\alpha}(x);S,V)dx = \inf_{\boldsymbol{\eta}\in\mathfrak{A}^p(\boldsymbol{a},\boldsymbol{\alpha};S,V)} \int_{\Omega\times V} f(\boldsymbol{e}_y(\boldsymbol{\eta}))dxdy.$$

$$\mathfrak{A}^p(\boldsymbol{a},\boldsymbol{\alpha};S,V){:=}\Big\{\boldsymbol{\eta}\in L^p(\Omega;W_0^{1,p}(V;\mathbb{R}^3)),\ \boldsymbol{\eta}(x,y)=\boldsymbol{a}(x)+\frac{2}{\mathrm{diam}S}\boldsymbol{\alpha}(x)\wedge\boldsymbol{y}\ \text{ in }\ \Omega\times S\Big\}.$$

If $(\boldsymbol{a},\boldsymbol{\alpha}) \in (C^\infty(\overline{\Omega};\mathbb{R}^3))^2$,

$$\int_\Omega \mathrm{cap}^f(\boldsymbol{a}(x),\boldsymbol{\alpha}(x);S,V)dx = \inf\left\{\int_{\Omega\times V} f(\boldsymbol{e}_y(\boldsymbol{\eta}))dxdy \,\middle|\, C^\infty(\overline{\Omega};\mathcal{D}(V;\mathbb{R}^3))\cap\mathfrak{A}^p(\boldsymbol{a},\boldsymbol{\alpha};S,V)\right\}.$$

Sobolev inequalities and Stokes paradox.

Minimum always achieved if $p < 2$ if we replace $W_0^{1,p}(V;\mathbb{R}^3)$ by a larger space.

$$\mathrm{cap}^f(\boldsymbol{a},\boldsymbol{\alpha};S,V) := \min\left\{ \int_V f(\boldsymbol{e}_y(\boldsymbol{\psi}))dy, \begin{array}{l} \boldsymbol{\psi} \in K_0^p(V;\mathbb{R}^3), \\ \boldsymbol{\psi}(y) = \boldsymbol{a} + \dfrac{2}{\mathrm{diam} S}\boldsymbol{\alpha} \wedge (\boldsymbol{y} - \boldsymbol{y}_S) \ \text{ in } S \end{array} \right\},$$

$$K_0^p(V;\mathbb{R}^3) := \overline{\mathcal{D}(V;\mathbb{R}^3)}^{|\cdot|_{K_0^p}}; \quad |\boldsymbol{\psi}|_{K_0^p} := \left(\int_V |\boldsymbol{\psi}|^{p^*} dy\right)^{\frac{1}{p^*}} + \left(\int_V |\nabla \boldsymbol{\psi}|^p dy\right)^{\frac{1}{p}}; \quad p^* := \frac{2p}{2-p}.$$

Proof: apply the Sobolev inequality $\left(\int_V |\boldsymbol{\psi}|^{p^*} dy\right)^{\frac{1}{p^*}} \leq C \left(\int_V |\nabla \boldsymbol{\psi}|^p dy\right)^{\frac{1}{p}}$
Consequence: $(\boldsymbol{a},\boldsymbol{\alpha}) \neq (0,0) \Rightarrow \mathrm{cap}^f(\boldsymbol{a},\boldsymbol{\alpha};S,\mathbb{R}^2) > 0$.

Minimum not achieved in general if $p \geq 2$ and V unbounded (Stokes' paradox).
$\mathrm{cap}^f(\boldsymbol{a},\boldsymbol{\alpha};S,V)$ may vanish for $(\boldsymbol{a},\boldsymbol{\alpha}) \neq (0,0)$ when V is unbounded.
$\mathrm{cap}^f(\boldsymbol{a},0;S,\mathbb{R}^2) = 0$ if $p = 2$,
$\mathrm{cap}^f(\boldsymbol{a},\boldsymbol{\alpha};S,\mathbb{R}^2) = 0$ if $p > 2$.

Asymptotic behavior of cap^f w.r.t. small sets. CASE $p < 2$.

Asymptotic study of $\mathrm{cap}^f(\boldsymbol{a}, 0; r_\varepsilon S, R_\varepsilon D)$ when $0 < r_\varepsilon \ll R_\varepsilon$

$$\lim_{\varepsilon\to 0} \frac{1}{r_\varepsilon^{2-p}} \mathrm{cap}^f(\boldsymbol{a}, \boldsymbol{\alpha}; r_\varepsilon S, R_\varepsilon D) = \mathrm{cap}^{f^{\infty,p}}(\boldsymbol{a}, \boldsymbol{\alpha}; S, \mathbb{R}^2).$$

$f^{\infty,p}(\boldsymbol{M}) := \limsup_{t\to+\infty} \frac{f(t\boldsymbol{M})}{t^p}$ p-recession function.

Proof: $\boldsymbol{M}$ large $\Rightarrow f^{\infty,p}(\boldsymbol{M}) \sim f(\boldsymbol{M})$.

$$\mathrm{cap}^f(\boldsymbol{a}, \boldsymbol{\alpha}; r_\varepsilon S, R_\varepsilon D) \sim \mathrm{cap}^{\overbrace{f^{\infty,p}}^{p\text{-homogeneous}}}(\boldsymbol{a}, \boldsymbol{\alpha}; r_\varepsilon S, R_\varepsilon D) = r_\varepsilon^{2-p} \underbrace{\mathrm{cap}^{f^{\infty,p}}(\boldsymbol{a}, \boldsymbol{\alpha}; S, \tfrac{R_\varepsilon}{r_\varepsilon} D)}_{\to \mathrm{cap}^{f^{\infty,p}}(\boldsymbol{a},\boldsymbol{\alpha};S,\mathbb{R}^2)}$$

Asymptotic behavior of cap^f w.r.t. small sets. CASE $p \geq 2$.

CASE $p = 2$, f quadratic. The sequences $(\mathrm{cap}^f(\boldsymbol{a}, 0; r_\varepsilon S, R_\varepsilon D))$ and $(\mathrm{cap}^f(0, \boldsymbol{\alpha}; r_\varepsilon S, R_\varepsilon D))$ show different orders of magnitude!

$$\lim_{\varepsilon \to 0} |\log r_\varepsilon| \mathrm{cap}^f(\boldsymbol{a}, 0; r_\varepsilon S, R_\varepsilon D) = \boldsymbol{a} \cdot \boldsymbol{A}^f \boldsymbol{a}, \quad \boldsymbol{A}^f \in \mathbb{S}^3_{>0} \quad \text{independent of } S,$$

$$\mathrm{cap}^f(0, \zeta \boldsymbol{e}_3; r_\varepsilon S, R_\varepsilon D) \geq C|\zeta|^2.$$

Proof: explicit computations in B. Gruais 05, B. 2012 if $f(\boldsymbol{M}) := \frac{\lambda_0}{2} \mathrm{tr}(\boldsymbol{M})^2 + \mu_0 |\boldsymbol{M}|^2$. Then use the proof of the homogenization result (tricky).

CASE $p > 2$.

$$\mathrm{cap}^f(\boldsymbol{a}, \zeta \boldsymbol{e}_3; r_\varepsilon S, R_\varepsilon D) \geq \frac{C}{R_\varepsilon^{p-2}} \left(|\boldsymbol{a}|^p + |\zeta|^p \right).$$

Application to homogenization

$\Omega := \Omega' \times (0, L)$ be a bounded Lipschitz cylindrical open subset of $\mathbb{R}^3$

$$(\mathcal{P}_\varepsilon) : \begin{cases} \min\limits_{\boldsymbol{u}_\varepsilon \in W_b^{1,p}(\Omega;\mathbb{R}^3)} F_\varepsilon(\boldsymbol{u}_\varepsilon) - \int_\Omega \boldsymbol{f}.\boldsymbol{u}_\varepsilon dx, \quad \boldsymbol{f} \in L^{p'}(\Omega;\mathbb{R}^3), \quad \left(\frac{1}{p} + \frac{1}{p'} = 1\right), \\ F_\varepsilon(\boldsymbol{u}_\varepsilon) := \int_{\Omega \setminus T_{r_\varepsilon}} f(\boldsymbol{e}(\boldsymbol{u}_\varepsilon))dx + l_\varepsilon \int_{T_{r_\varepsilon}} g(\boldsymbol{e}(\boldsymbol{u}_\varepsilon))dx, \quad \boldsymbol{e}(\boldsymbol{u}_\varepsilon) = \frac{1}{2}(\boldsymbol{\nabla} \boldsymbol{u}_\varepsilon + \boldsymbol{\nabla}^T \boldsymbol{u}_\varepsilon), \\ W_b^{1,p}(\Omega;\mathbb{R}^3) = \left\{ \boldsymbol{\psi} \in W^{1,p}(\Omega;\mathbb{R}^3),\ \boldsymbol{\psi} = 0 \ \text{ on } \Omega' \times \{0\} \right\}. \end{cases}$$

- f, g strictly convex. $\quad c|\boldsymbol{M}|^p \leq f(\boldsymbol{M}), g(\boldsymbol{M}) \leq C|\boldsymbol{M}|^p \quad \forall \boldsymbol{M} \in \mathbb{S}^3, \quad 1 < p < +\infty.$

-$l_\varepsilon \gg 1$.

- simplified model of small deformation nonlinear elasticity ($\boldsymbol{u}_\varepsilon$: displacement), or Norton-Hoff model of viscoplasticity ($\boldsymbol{u}_\varepsilon$: velocity).

The distribution of fibers T_{r_ε}

T_{r_ε} : ε-periodic set of parallel disjoint cylinders ($\varepsilon \to 0$) of cross sections of size $r_\varepsilon << \varepsilon$ homothetical to some bounded Lipschitz domain S of $\mathbb{R}^2$.

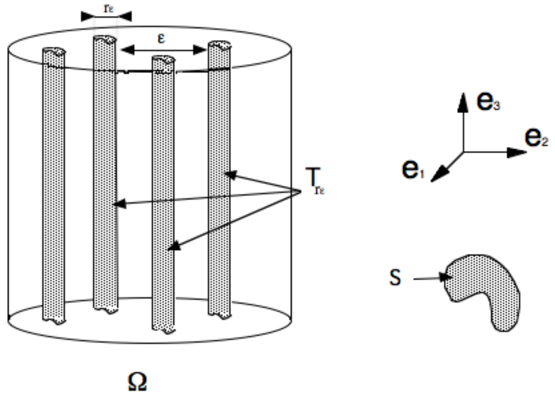


Figure: the fiber reinforced composite

The capacity density $c^f(\boldsymbol{a}, \zeta)$

The limit problem depends on the pointwise limit $c^f(\boldsymbol{a}, \zeta)$ of the sum per unit surface of the images under $\mathrm{cap}^f(\boldsymbol{a}, \zeta \boldsymbol{e}_3; ., .)$ of the cross sections of the fibers with respect to suitably larger sets

$$c^f(\boldsymbol{a}, \zeta) := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \mathrm{cap}^f(\boldsymbol{a}, \zeta \boldsymbol{e}_3; r_\varepsilon S, R_\varepsilon D), \qquad r_\varepsilon \ll R_\varepsilon \ll \varepsilon.$$

Setting

$$\gamma^{(p)} := \lim_{\varepsilon \to 0} \gamma_\varepsilon^{(p)}(r_\varepsilon); \quad \gamma_\varepsilon^{(p)}(r) := \frac{r^{2-p}}{\varepsilon^2} \text{ if } p \neq 2; \quad \gamma_\varepsilon^{(p)}(r) := \frac{1}{\varepsilon^2 |\log r|} \text{ if } p = 2,$$

we obtain

$$\begin{cases} c^f(\boldsymbol{a}, \zeta) := \gamma^{(p)} \mathrm{cap}^{f^{\infty, p}}(\boldsymbol{a}, \zeta \boldsymbol{e}_3; S, \mathbb{R}^2), & \text{if } 1 < p < 2, \\ c^f(\boldsymbol{a}, \zeta) := \gamma^{(2)} \boldsymbol{a} \cdot \boldsymbol{A}^f \boldsymbol{a} \text{ if } \zeta = 0, \quad c^f(\boldsymbol{a}, \zeta) := +\infty \text{ otherwise}, & \text{if } p = 2, \\ c^f(\boldsymbol{a}, \zeta) := 0 \text{ if } (\boldsymbol{a}, \zeta) = (0, 0), \quad c^f(\boldsymbol{a}, \zeta) := +\infty \text{ otherwise}, & \text{if } p > 2. \end{cases}$$

Critical case: $0 < \gamma^{(p)} < +\infty$

Occurs when $p < 2$ and $\frac{r_\varepsilon^{2-p}}{\varepsilon^2}$ of order 1 or $p = 2$ and $\frac{1}{\varepsilon^2 |\log r_\varepsilon|}$ of order 1.

$\Rightarrow$ discrepancy between the displacements in the fibers and in the matrix.

Displacement in the fibers described in terms of $(\boldsymbol{v}, \theta)$ weak limits in L^p of

$$\boldsymbol{v}_\varepsilon(x) := \sum_{i \in I_\varepsilon} \left(\fint_{S^i_{r_\varepsilon}} \boldsymbol{u}_\varepsilon(s, x_3) d\mathcal{H}^2(s) \right) \mathbb{1}_{Y^i_\varepsilon}(x') \qquad (=: \overline{\mathfrak{v}}^S_\varepsilon(\boldsymbol{u}_\varepsilon)(x)),$$

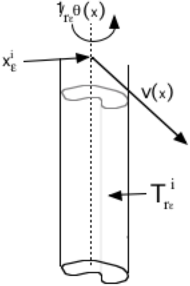
$$\theta_\varepsilon(x) := \sum_{i \in I_\varepsilon} \frac{1}{c} \fint_{S^i_{r_\varepsilon}} \left(-\frac{y_{\varepsilon 2}}{r_\varepsilon} u_1 + \frac{y_{\varepsilon 1}}{r_\varepsilon} u_2 \right)(s, x_3) d\mathcal{H}^2(s) \mathbb{1}_{Y^i_\varepsilon}(x') \qquad (= \overline{\theta}^S_\varepsilon(\boldsymbol{u}_\varepsilon)(x)),$$

$$c := \frac{2 \fint_S |y|^2 dy}{\mathrm{diam} S},$$

where $\boldsymbol{u}_\varepsilon$ solution to $(\mathcal{P}_\varepsilon)$.

Concentration of energy around the fibers

Displacement in the fibers $T^i_{r_\varepsilon}$: $\boldsymbol{u}_\varepsilon(x) \sim \boldsymbol{v}(x) + \frac{1}{r_\varepsilon}\theta \boldsymbol{e}_3 \wedge \left(\boldsymbol{x} - \boldsymbol{x}^i_\varepsilon\right)$.



Concentration of energy in a small outer neighborhood of the fibers:

$$\int_\Omega c^f(\boldsymbol{v} - \boldsymbol{u}, \theta)dx \qquad \boldsymbol{u} := \text{ weak limit of } \boldsymbol{u}_\varepsilon \text{ in } W^{1,p}_0.$$

Concentration of energy in the fibers. Case $0 < k < +\infty$

Parameters k and κ:

$$k := \lim_{\varepsilon\to 0} k_\varepsilon \in]0,+\infty]; \qquad \kappa := \lim_{\varepsilon\to 0} r_\varepsilon^p k_\varepsilon \in [0,+\infty]; \qquad k_\varepsilon := l_\varepsilon \frac{r_\varepsilon^2 |S|}{\varepsilon^2}.$$

- $k < +\infty$: combination of extensional and torsional strain energies:

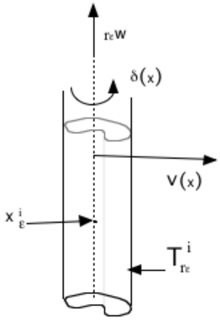
$$\int_\Omega g^{hom}\left(\frac{\partial v_3}{\partial x_3}, \frac{\partial \theta}{\partial x_3}\right) dx,$$

where

$$\begin{cases} g^{hom} : (a,\beta) \in \mathbb{R}^2 \to \inf\limits_{\boldsymbol{q}\in C^\infty(\overline{S};\mathbb{R}^3)} k \fint_S g\left(\mathfrak{L}(\boldsymbol{q},a,\beta)\right) dy, \\ \mathfrak{L}(\boldsymbol{q},a,\beta) := \boldsymbol{e}_y(\boldsymbol{q}) + \dfrac{2}{\mathrm{diam} S}\beta(-y_2\boldsymbol{e}_1 \odot \boldsymbol{e}_3 + y_1\boldsymbol{e}_2 \odot \boldsymbol{e}_3) + a\boldsymbol{e}_3 \otimes \boldsymbol{e}_3. \end{cases}$$

Case $0 < \kappa < +\infty$**.**

-Extra variables w, δ= weak limits in L^p of $\left(\frac{v_{\varepsilon 3}}{r_\varepsilon}\right)$ and $\left(\frac{\theta_\varepsilon}{r_\varepsilon}\right)$.
Displacement in the fibers $T^i_{r_\varepsilon}$:
$\boldsymbol{u}_\varepsilon(x) \sim v_1(x)\boldsymbol{e}_1 + v_2(x)\boldsymbol{e}_2 + r_\varepsilon w(x)\boldsymbol{e}_3 + \delta\boldsymbol{e}_3 \wedge \left(\boldsymbol{x} - \boldsymbol{x}^i_\varepsilon\right)$.



Concentration of energy in the fibers when $0 < \kappa < +\infty$

Combination of torsional, bending, and extensional strain energies:

$$\int_\Omega g^{hom}\left(\frac{\partial^2 v_1}{\partial x_3^2}, \frac{\partial^2 v_2}{\partial x_3^2}, \frac{\partial w}{\partial x_3}, \frac{\partial \delta}{\partial x_3}\right) dx,$$

where

$$g^{hom} : (\zeta_1, \zeta_2, a, \beta) \in \mathbb{R}^4 \to \inf_{\boldsymbol{q} \in C^\infty(\overline{S};\mathbb{R}^3)} \kappa \fint_S g^{0,p}\left(\mathfrak{J}(\boldsymbol{q}, \zeta_1, \zeta_2, a, \beta)\right) dy,$$

$$\mathfrak{J}(\boldsymbol{q}, \zeta_1, \zeta_2, a, \beta) := \mathfrak{L}(\boldsymbol{q}, a, \beta) - \left(\sum_{\alpha=1}^{2} \zeta_\alpha y_\alpha\right) \boldsymbol{e}_3 \otimes \boldsymbol{e}_3,$$

$$g^{0,p}(\boldsymbol{M}) := \liminf_{t \to 0^+} \frac{g(t\boldsymbol{M})}{t^p}.$$

Homogenisation result

Theorem

If $\gamma^{(p)} > 0$, $(\boldsymbol{u}_\varepsilon, \boldsymbol{v}_\varepsilon, \theta_\varepsilon) \to (\boldsymbol{u}, \boldsymbol{v}, \theta)$ *weakly in* $W^{1,p}(\Omega; \mathbb{R}^3) \times L^p(\Omega; \mathbb{R}^3) \times L^p(\Omega)$ *and, if* $0 < \kappa < +\infty$, $(\frac{1}{r_\varepsilon} v_{\varepsilon 3}, \frac{1}{r_\varepsilon}\theta_\varepsilon) \to (w, \delta)$ *weakly* $(L^p(\Omega))^2$ *to* (w, δ)*, where, setting*

$$\boldsymbol{v}^{tuple} := (\boldsymbol{v}, \theta) \quad P(\boldsymbol{v}, \theta) := \left(\frac{\partial v_3}{\partial x_3}, \frac{\partial \theta}{\partial x_3}\right) \quad \text{if } \kappa = 0,$$

$$\boldsymbol{v}^{tuple} := (\boldsymbol{v}, \theta, w, \delta) \quad P(\boldsymbol{v}, \theta, w, \delta) = \left(\frac{\partial^2 v_1}{\partial x_3^2}, \frac{\partial^2 v_2}{\partial x_3^2}, \frac{\partial w}{\partial x_3}, \frac{\partial \delta}{\partial x_3}\right) \text{if } 0 < \kappa \leq +\infty,$$

$(\boldsymbol{u}, \boldsymbol{v}^{tuple})$ *is the unique solution to*

$$(\mathcal{P}^{hom}) : \min_{(\boldsymbol{u}, \boldsymbol{v}^{tuple}) \in W_b^{1,p}(\Omega; \mathbb{R}^3) \times \mathcal{D}} \Phi(\boldsymbol{u}, \boldsymbol{v}^{tuple}) - \int_\Omega \boldsymbol{f}.\boldsymbol{u} dx.$$

$$\Phi(\boldsymbol{u}, \boldsymbol{v}^{tuple}) := \int_\Omega f(\boldsymbol{e}(\boldsymbol{u}))dx + c^f(\boldsymbol{v} - \boldsymbol{u}, \theta) + g^{hom}(P(\boldsymbol{v}^{tuple}))dx,$$

The domain $\mathcal{D}$

$\mathcal{D}$ characterizes the boundary and regularity conditions satisfied by $(\boldsymbol{v}^{tuple})$:

$$\mathcal{D} := \Big\{(\boldsymbol{v},\theta) \in L^p(\Omega;\mathbb{R}^3) \times L^p(\Omega'; W^{1,p}(0,L)),\\ v_3 \in L^p(\Omega'; W^{1,p}(0,L)), v_3 = \theta = 0 \text{ on } \Omega' \times \{0\}\Big\} \text{ if } 0 < k < +\infty,$$

$$\mathcal{D} := \Big\{(\boldsymbol{v},\theta) \in L^p(\Omega;\mathbb{R}^3) \times L^p(\Omega),\ v_3 = \theta = 0\Big\} \qquad \text{if } (k,\kappa) = (+\infty, 0),$$

$$\mathcal{D} := \Big\{(\boldsymbol{v},\theta,w,\delta) \in L^p(\Omega'; W^{2,p}(0,L;\mathbb{R}^3)) \times L^p(\Omega'; W^{1,p}(0,L))^3, v_3 = \theta = 0,\\ w = \delta = v_\alpha = \frac{\partial v_\alpha}{\partial x_3} = 0 \text{ on } \Omega' \times \{0\}\ (\alpha \in \{1,2\})\Big\} \text{ if } 0 < \kappa < +\infty,$$

$$\mathcal{D} := \{(0,0,0,0)\} \qquad \text{if } k = \kappa = +\infty.$$

Non-periodic distribution of fibers.

$G_\varepsilon \subset \Omega'$, G_ε finite, arbitrary

$$\sup_{\varepsilon>0,\ \boldsymbol{i}\in\mathbb{Z}^2} \sharp\, G_\varepsilon \cap (\varepsilon\boldsymbol{i} + \varepsilon Y) < +\infty, \qquad \min_{\boldsymbol{\omega},\boldsymbol{\omega}'\in G_\varepsilon,\boldsymbol{\omega}\neq\boldsymbol{\omega}'} |\boldsymbol{\omega} - \boldsymbol{\omega}'| > d_\varepsilon \gg r_\varepsilon,$$

$$T_{r_\varepsilon}(G_\varepsilon) := \bigcup_{\boldsymbol{\omega}\in G_\varepsilon} T^{\boldsymbol{\omega}}_{r_\varepsilon}, \quad T^{\boldsymbol{\omega}}_{r_\varepsilon} := S^{\boldsymbol{\omega}}_{r_\varepsilon} \times (0, L), \quad S^{\boldsymbol{\omega}}_{r_\varepsilon} := (\boldsymbol{\omega} + r_\varepsilon S),$$

$(\mathcal{P}_\varepsilon(G_\varepsilon))$: deduced from $(\mathcal{P}_\varepsilon)$ by substituting $T_{r_\varepsilon}(G_\varepsilon)$ for T_{r_ε}.

$$n_{G_\varepsilon}(x) := \sum_{\boldsymbol{i}\in\mathbb{Z}^2,\ \varepsilon\boldsymbol{i}+\varepsilon Y\subset\Omega'} \sharp\,(G_\varepsilon \cap (\varepsilon\boldsymbol{i} + \varepsilon Y))\ \mathbb{1}_{\varepsilon\boldsymbol{i}+\varepsilon Y}(x') \overset{L^\infty(\Omega)}{\rightharpoonup} n,$$

Theorem

The solution to $(\mathcal{P}_\varepsilon(G_\varepsilon))$ weakly converges in $W^{1,p}_b(\Omega;\mathbb{R}^3)$ to the unique solution to

$$(\mathcal{P}^{hom,\,n}) : \min_{(\boldsymbol{u},\boldsymbol{v}^{tuple})\in W^{1,p}_b(\Omega)\times\mathcal{D}^{(n)}} \Phi^{(n)}(\boldsymbol{u},\boldsymbol{v}^{tuple}) - \int_\Omega \boldsymbol{f}.\boldsymbol{u}dx,$$

where

$$\Phi^{(n)}(\boldsymbol{u},\boldsymbol{v}^{tuple}) = \int_\Omega f(\boldsymbol{e}(\boldsymbol{u}))dx + \int_\Omega c^f(\boldsymbol{v} - \boldsymbol{u},\theta)ndx + \int_\Omega g^{hom}(P(\boldsymbol{v}^{tuple}))ndx.$$

Random distribution of fibers.

$$\mathfrak{O} := \left\{\omega \in 2^{\mathbb{R}^2},\ \forall(\omega_1,\omega_2) \in \omega^2,\ \omega_1 \neq \omega_2 \ \Rightarrow\ |\omega_1 - \omega_2| \geq d\right\},$$

$$G_\varepsilon(\omega) := \varepsilon\omega \cap \{x \in \Omega',\ \mathrm{dist}(x, \partial\Omega') \geq 5\sqrt{2}\varepsilon\}$$

- $\mathcal{B}_{\mathfrak{O}}$ the Borel σ-algebra generated by the Hausdorff distance on $\mathfrak{O}$.
-P probability on $(\mathfrak{O}, \mathcal{B}_{\mathfrak{O}})$ such that $P(A+z) = P(A)\ \forall\, z \in \mathbb{Z}, \forall\, A \in \mathcal{B}_{\mathfrak{O}}$
- $\mathcal{F}$ σ-algebra of the Y-periodic elements of $\mathcal{B}_{\mathfrak{O}}$
- $n_0 : \omega \in \mathfrak{O} \to n_0(\omega) := \sharp\left(\omega \cap \left[-\frac{1}{2}, \frac{1}{2}\right[^2\right)$
-$E_P^{\mathcal{F}} n_0$ conditional expectation of n_0 given $\mathcal{F}$ with respect to P.

Theorem

There exists $(\varepsilon_k)_{k\in\mathbb{N}}$ and $\mathfrak{N} \in \mathcal{B}_{\mathfrak{O}}$ with $P(\mathfrak{N}) = 0$ such that, for all $\omega \in \mathfrak{O} \setminus \mathfrak{N}$, the solution $\boldsymbol{u}_{\varepsilon_k}(\omega)$ to $(\mathcal{P}_{\varepsilon_k}(G_{\varepsilon_k}(\omega)))$ weakly converges in $W_b^{1,p}(\Omega; \mathbb{R}^3)$ to the unique solution $\boldsymbol{u}(\omega)$ to $(\mathcal{P}^{hom,\ E_P^{\mathcal{F}} n_0(\omega)})$.

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