Non-classical homogenisation, related analytic tools and applications to dynamic problems with partially high contrasts.

Shane Cooper

18th September 2008

Abstract

Two main objectives have been fulfilled in this project. Firstly, the study of the required background material and relevant topics in analysis used in classical and non-classical homogenisation. This was followed by the study of recent work and development of new tools on the propagation and localisation of elastic waves in highly anisotropic periodic composites.

Acknowledgements

I would like to take this opportunity to thank my supervisors Professor Valery P. Smyshlyaev and Dr. Ilia Kamotski for their continued support and endless source of inspiration. Special thanks are also due to Dr Nicolas Dirr for the provision of a set of lecture notes on Measure Theory.

Declaration

I certify that all material in this dissertation is my own work, except where I have indicated with appropriate references.

....
Shane Alan Lee Cooper
Department of Mathematical Sciences
University of Bath

18/09/2008

Contents

List of Figures

Introduction

It is known that the behaviour of a composite material is often different to the behaviour of its constituent parts, reinforcing or in some cases exhibiting different behaviours to that of its independent constituents. It is desirable to know how the behaviour of a composite material is affected by the constituents that make it. This leads to (even in the simplest of cases) complicated boundary value problems for partial differential equations (PDE's) that cannot be solved directly by analytical or numerical means. Let us show this with an example. Consider the following problem

$$-\text{ div } (a(x)\nabla u(x)) = f(x), \quad \text{in } \Omega \tag{1}$$

$$u = 0, \quad \text{on } \partial\Omega. \tag{2}$$

This equation is known to be the governing equation of many physical effects, such as for example heat flow through a body Ω. In this case f is a heat source, $u(x)$ is the temperature at a given point $x \in \Omega$ and $a(x)$ is the thermal conductivity of the body Ω. Other physical examples which this system of equations governs include electrical conductivity, linear elasticity, electrostatics, etc.

Now if we were to add periodically throughout the material a second material (with a different thermal conductivity) such that no two pieces of this second material where less than ε apart, we would have produced a composite material with the temperature at any point in the composite being governed by (1)-(2). Now the thermal conductivity is described by

$$a(x) = \begin{cases} \gamma_1 & \text{in component one} \\ \gamma_2 & \text{in component two.} \end{cases}$$

As we can see the coefficient to the PDE depends on the parameter ε and if ε is small, which is the case for composite materials, then the coefficient will be rapidly oscillating.

Homogenisation theory has been developed over the last few decades to deal with problems of this form, that is to find a way of tackling PDE's with rapidly oscillating coefficients. If the coefficients of a PDE depend on a small parameter ε, where u^ε is the solution of the PDE, then by taking a sequence of positive ε that tend to zero we will produce a corresponding sequence of solutions $\{u^\varepsilon\}$. The aim of Homogenisation theory is to answer the following questions: Does this sequence $\{u^\varepsilon\}$ converge to some limit u^0 as $\varepsilon \to 0$? If so then does there exist some 'limit PDE', which admits u^0 as a solution, and are the coefficients of this PDE independent of ε? If this limit u^0 exists, how well does it approximate the original solution u^ε?

The "classical" homogenisation is used to study heterogeneous materials with moderate contrasts in its physical properties. In classical homogenisation, the homogenised limit solution approximates the original solution well but does not account for a number of microscopic effects that may be present. The presence of high-contrasts between the coefficients in the matrix and inclusion has been shown to account for some of these effects. For the heterogeneous medium by choosing the coefficients within the 'inclusion' to be of the order of a small parameter δ, then sending δ and ε to zero is known as high-contrast homogenisation. High-contrast problems require the development and tools of "non-classical" homogenisation.

In Section 1 we shall review the results of classical and high-contrast homogenisation for a periodic isotropic composite material. First the classical result shall be stated and then the high-contrast result shall be derived by the method of asymptotic expansion. These results will then be applied to the wave equation and we shall see formally how waves of certain frequencies do not propagate through the material, a phenomenon know as the 'band-gap' effect, to be present in the case of high-contrasts. Section 2 is dedicated to mathematical justification of the homogenisation problem, reviewing the functional spaces that are important to analysis of partial differential equations, highlighting how in the case of classical homogenisation for periodic composites the asymptotic expansion can be formally justified with the use of the Lax-Milgram lemma. The method of two-scale convergence is also reviewed here, along with an example of how two-scale convergence is used in the case of high contrasts. Section 3 expands on the recent work in [1] with the consideration to the specific case of isolated inclusions. In this section we find the limit solution for the elastodynamic equations of motion for a more general class of partially high-contrasting elasticity tensor C^{ε}. Then we shall study the example of spherical inclusions consisting of an isotropic elastic material (in $\mathbb{R}^3$) with Lamé coefficients $\mu \sim O(\varepsilon^2)$ and $\lambda \sim O(1)$ in $\mathbb{R}^3$. This section ends with a look at the mathematical justification of the solution to the unit cell problem for this particular example. In Section 4 we give a brief review of the important properties of Lebesgue integration. Finally we finish with a discussion of possible further developments of the work present in section 3.

Notation

Throughout this thesis we shall adopt the Einstein summation convention, i.e. we sum over repeated indices. For example $n_i a_{ij} n_j = \sum_{i,j} n_i a_{ij} n_j$.

$u_{,i}(x)$ denotes the partial derivative of u with respect to the x_ith variable, i.e. $\frac{\partial u}{\partial x_i}$.

X any linear space.

$\|x\|_X$ the norm of $x \in X$, when X is a normed space.

X^* the dual space of X, i.e. the space of linear continuous functionals on X.

(u, v) denotes the inner product between $u, v \in X$, when X is an inner product space.

$\langle F, v\rangle$ denotes the image $F(v)$ for $v \in X$ and $F \in X^*$.

$C_0^\infty(\Omega)$ denotes the space of infinitely smooth functions with compact support in Ω.

$C_\#^\infty(Q)$ denotes the subspace of $C^\infty(\mathbb{R}^n)$ of Q-periodic functions.

$f\big|_i$ denotes the value of the function f in Q_i, for $i = 1, 2$.

$\mathcal{D}[\Omega; C_\#^\infty(Q)]$ denotes the space of functions on $\Omega \times \mathbb{R}$ such that $u(x, \cdot) \in C_\#^\infty(Q)$ for any $x \in \Omega$ and the map $x \mapsto u(x, \cdot) \in C_\#^\infty(Q)$ is infinitely differentiable with compact support in Ω.

$L^p(\Omega; X)$ denotes the set of measurable functions $u : x \in \Omega \mapsto u(x) \in X$, where X is a Banach space, such that $\|u(x)\|_X \in L^p(\Omega)$.

Chapter 1

Classical and Non-classical homogenisation for an isotropic medium

1.1 Problem formulation

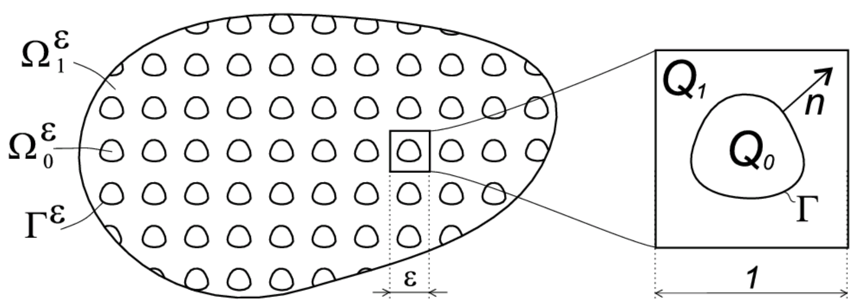

Figure 1.1: The periodic geometry and a periodicity cell Q with isolated inclusions

Let us consider (see Figure 1.1) a heterogeneous material occupying Ω with periodic isolated inclusions. Let $Q_0 \subset Q$ be the inclusion and $Q_1 = Q \backslash Q_0$ the matrix, with $\Gamma = Q_0 \cap Q_1$ being the smooth boundary between matrix and inclusion. Setting $\chi_i(y)$ for $i = 0, 1$ to be characteristic function of Q_i extended by periodicity to $\mathbb{R}^n$, Ω is divided into two subdomains Ω_1^ε and Ω_0^ε:

$$\Omega_0^\varepsilon = \left\{ x \in \Omega \mid \chi_0\left(\tfrac{x}{\varepsilon}\right) = 1 \right\}, \quad \Omega_1^\varepsilon = \left\{ x \in \Omega \mid \chi_1\left(\tfrac{x}{\varepsilon}\right) = 1 \right\}.$$

$\Gamma^\varepsilon = \Omega_1^\varepsilon \bigcap \Omega_0^\varepsilon$.

1.2 Classical homogenisation

Let us consider the following linear second-order elliptic problem

$$- \operatorname{div}\left(a\left(\frac{x}{\varepsilon}\right)\nabla u^{\varepsilon}\right) = f^{\varepsilon}(x), \quad \text{in } \Omega \tag{1.1}$$

$$u^{\varepsilon} = 0, \quad \text{on } \partial\Omega. \tag{1.2}$$

where $a^{\varepsilon}(x) = a\left(\frac{x}{\varepsilon}\right)$. Let us assume $a(y)$ is strictly positive definite, $a(y) \geq \nu, \forall y, \nu > 0$ and Q-periodic with period 1, $f^{\varepsilon}(x) = f\left(x, \frac{x}{\varepsilon}\right)$ is allowed to be locally varying. It is well known and easy to show (see section 2.3) that the limit problem of (1.1)-(1.2) is

$$- \operatorname{div}_x\left(\hat{a}\nabla u^0(x)\right) = F(x), \quad \text{in } \Omega \tag{1.3}$$

$$u^0 = 0, \quad \text{on } \partial\Omega, \tag{1.4}$$

where $\hat{a} = \langle a(y)\left(\nabla_y N + I\right)\rangle_Q$ is the homogenised matrix, written in component form $\hat{a}_{ij} = \left\langle a(y)\left(\frac{\partial N_j}{\partial y_i} + \delta_{ij}\right)\right\rangle_Q$ and $F(x) = \langle f(x,y)\rangle$. $N_j(y)$ for $j = 1, 2, \ldots, n$ is the Q-periodic solution to the following problem:

$$-\operatorname{div}_y\left(a(y)\nabla_y N_j(y)\right) = \frac{\partial a}{\partial y_j}, \quad \text{in } Q \tag{1.5}$$

$$\langle N_j(y)\rangle_Q = 0. \tag{1.6}$$

Here and anywhere else in the paper we shall use the following definition

$$< f(\cdot, y) >_{\Omega} := \int_{\Omega} f(\cdot, y)\ \mathrm{dy}, \quad y \in Q. \tag{1.7}$$

Notice that $\hat{a}$ is a constant matrix and the limit solution u^0 depends on x only.

1.3 Non-classical homogenisation

We shall now consider the linear second-order elliptic problem (1.1)-(1.2) for a heterogeneous material with highly contrasting inclusions

$$a(y) = \begin{cases} 1, & y \in Q_1 \\ \varepsilon^2, & y \in Q_0. \end{cases}$$

With the following boundary conditions which are imposed on the internal boundary Γ

$$u^{\varepsilon}|_1 = u^{\varepsilon}|_0 \tag{1.8}$$

$$a\left(\frac{x}{\varepsilon}\right)\frac{\partial u^{\varepsilon}}{\partial n}\bigg|_1 = a\left(\frac{x}{\varepsilon}\right)\frac{\partial u^{\varepsilon}}{\partial n}\bigg|_0, \tag{1.9}$$

which can be physically understood, in the context of linear elasticity for example, as the continuity of deformation and traction across the interface between the two materials.

Let us seek an asymptotic solution of the form

$$u^\varepsilon(x) = u^{(0)}\left(x, \frac{x}{\varepsilon}\right) + \varepsilon u^{(1)}\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^2 u^{(2)}\left(x, \frac{x}{\varepsilon}\right) + O(\varepsilon^3),$$

where $u^{(0)}(x,y), u^{(1)}(x,y)$ and $u^{(2)}(x,y)$ are Q-periodic in y. From substitution into (1.1) we arrive at the following system of equations

$$-\Delta_y u^{(0)} = 0, \qquad y \in Q_1, \tag{1.10}$$
$$-\Delta_y u^{(0)} = f(x,y), \qquad y \in Q_0, \tag{1.11}$$
$$-\Delta_y u^{(1)} = 0, \qquad y \in Q_1, \tag{1.12}$$
$$-\Delta_y u^{(2)} = f(x,y) + 2\nabla_x \cdot \nabla_y u^{(1)} + \Delta_x u^{(0)}, \qquad y \in Q_1. \tag{1.13}$$

With substitution of the asymptotic expansion into the boundary conditions (1.2),(1.8) & (1.9) results in the following boundary conditions

$$u^{(0)}\Big|_1 = u^{(0)}\Big|_0, \qquad y \in \Gamma, \tag{1.14}$$
$$n_i \frac{\partial u^{(0)}}{\partial y_i}\Bigg|_1 = 0, \qquad y \in \Gamma, \tag{1.15}$$
$$n_i \frac{\partial u^{(0)}}{\partial x_i}\Bigg|_1 + n_i \frac{\partial u^{(1)}}{\partial y_i} = 0, \qquad y \in \Gamma, \tag{1.16}$$
$$n_i \frac{\partial u^{(1)}}{\partial x_i} + n_i \frac{\partial u^{(2)}}{\partial y_i} = n_i \frac{\partial u^{(0)}}{\partial y_i}\Bigg|_0, \qquad y \in \Gamma. \tag{1.17}$$

Multiplying (1.10) by $u^{(0)}$, integrating over the region Q_1, then applying integration by parts and equation (1.15) we see that $\nabla_y u^{(0)} = 0$ for $y \in Q_1$ which implies $u^{(0)}(x,y) = u_0(x)$ in Q_1. This result and equation (1.11) lead to the assumption that u^0 has the form

$$u^{(0)}(x,y) = \begin{cases} u_0(x) & y \in Q_1 \\ u_0(x) + v(x,y) & y \in Q_0. \end{cases} \tag{1.18}$$

Note this form is formally justified with the use of two-scale convergence (see Section 2.4). Upon substitution of (1.18) into (1.11) and (1.14), we see $v(x,y)$ is the solution to the following system of equations

$$-\Delta_y v(x,y) = f(x,y), \quad y \in Q_0 \tag{1.19}$$
$$v(x,y) = 0, \qquad y \in \Gamma. \tag{1.20}$$

Seeking a solution of the form $u^{(1)}(x,y) = N_j(y)\frac{\partial u_0}{\partial x_j}$, where N_j is Q periodic, from (1.12) and (1.16) we see that N_j, for $j = 1, \ldots, n$, is a solution to the following system of equations

$$-\Delta N_j = 0, \quad y \in Q_1 \tag{1.21}$$
$$n_i \left(\frac{\partial N_j}{\partial y_i} + \delta_{ij}\right) = 0, \quad y \in \Gamma. \tag{1.22}$$

Using Green's second identity

$$\int_Q (v\Delta u - u\Delta v)\,\mathrm{dx} = \int_{\partial\Omega} \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n} \right) \mathrm{dS}$$

for $u = u^{(2)}(x,y)$, $v = 1$ in the domain Q_1, along with (1.13) and (1.17), gives us:

$$\int_{Q_1} \left(f(x,y) + 2\frac{\partial^2 u^1}{\partial x_i \partial y_i} + \frac{\partial^2 u^0}{\partial x_i^2} \right) \mathrm{dy} = -\int_\Gamma \left(n_i \frac{\partial v}{\partial y_i} - n_i \frac{\partial u^1}{\partial x_i} \right) \mathrm{dS}, \tag{1.23}$$

we see from (1.11) and applying Green's second identity to $u = 1$, in Q_0, that

$$\int_\Gamma n_i \frac{\partial v}{\partial y_i} \mathrm{dS} = \int_{Q_0} \Delta v \mathrm{dy} = \int_{Q_0} f(x,y)\mathrm{dy}.$$

Also notice via divergence theorem

$$\int_\Omega \nabla \cdot F \ \mathrm{dx} = \int_{\partial\Omega} F \cdot n \ \mathrm{dS}$$

that

$$\int_\Gamma n \cdot \nabla_x u^1 \mathrm{dS} = \int_\Gamma n \cdot \left(N_j \nabla_x \left(\frac{\partial u_0}{\partial x_j} \right) \right) \mathrm{dS} = \int_{Q_1} \nabla_y \cdot \left(N_j \nabla_x \left(\frac{\partial u_0}{\partial x_j} \right) \right) \mathrm{dy} = \int_{Q_1} \frac{\partial N_j}{\partial y_i} \frac{\partial^2 u_0}{\partial x_i \partial x_j} \mathrm{dy}.$$

Therefore (1.23) reduces to the following boundary value problem, which is the homogenised problem

$$-\mathrm{div}\left(A^{\mathrm{hom}} \nabla u_0 \right) = < f(x,y) >_Q, \quad \text{in } \Omega \tag{1.24}$$

$$u_0 = 0, \qquad \text{on } \partial\Omega \tag{1.25}$$

where $A^{\mathrm{hom}}_{ij} = \int_{Q_1} \left(\frac{\partial N_j}{\partial y_i} + \delta_{ij} \right) \mathrm{dy}$.

As we can see the main difference here to the classical homogenisation problem is that the limit solution u^0 retains information about the fast variable y as well as the slow variable x. Looking at the form of the limit solution u^0, we see can see that u_0 is the slowly varying 'homogenised average' of the original solution as captured in classical homogenisation, while $v(x,y)$ describes the rapid oscillations of the solution within the inclusion. These "microresonances" are unique to non-classical homogenisation and (as we will show) account for phenomenon not described by classical homogenisation. Section 2.4 gives a formal derivation of the homogenised limit problem for the high-contrast problem via the method of two-scale convergence.

1.4 Wave propagation

Let us consider the homogenisation problem for the wave equation:

$$\rho \ddot{u}^\varepsilon - \mathrm{div}\left(a\left(\frac{x}{\varepsilon} \right) \nabla u^\varepsilon \right) = f^\varepsilon(x,t), \quad \text{in } \Omega \tag{1.26}$$

$$u^\varepsilon = 0, \qquad \text{on } \partial\Omega \tag{1.27}$$

here we assume for simplicity that $\rho > 0$ is constant. Let $u^\varepsilon(x,t)$ be identically zero for $t < 0$, $f^\varepsilon(x,t) = f\left(x, \frac{x}{\varepsilon}, t\right)$, where $f^\varepsilon(x,t) \equiv 0$ for $t < 0$. Also $\dot{}$ represents differentiation with respect to time. Seeking a time harmonic solution, i.e. a solution of the form $u^\varepsilon(x,t) = e^{-i\omega t}u^\varepsilon(x)$, ω being the frequency, the problem (1.26)-(1.27) then becomes

$$-\mathrm{div}\left(a\left(\frac{x}{\varepsilon}\right)\nabla u^\varepsilon\right) = \lambda u^\varepsilon, \quad \text{in } \Omega \tag{1.28}$$

$$u^\varepsilon = 0, \qquad \text{on } \partial\Omega \tag{1.29}$$

where $\lambda = \rho\omega^2$. As we can see this is a linear second order elliptic Dirichlet problem of the form (1.1)-(1.2) with $f^\varepsilon(x)$ replaced λu^ε. Upon deriving the homogenised equations for the classical and high-contrast homogenisation problems, we will show that the presence of the microscopic resonances $v(x,y)$ lead to band-gap effects being formally shown in the high-contrast case but not present in the classical setting.

Let us consider (1.28)-(1.29) in the classical setting described in section 1.2, we arrive at the following limit problem

$$-\mathrm{div}_x\left(\hat{a}\nabla u^0(x)\right) = \lambda u^0(x), \quad \text{in } \Omega \tag{1.30}$$

$$u^0 = 0, \quad \text{on } \partial\Omega. \tag{1.31}$$

Let us seek a plane wave solution of the form $u^0(x) = Ae^{(ikx\cdot n)}$, where A is the amplitude, k is the wave number and n is the unit vector describing the propagation direction. Upon simple substitution into (1.30)-(1.31) we arrive at

$$k^2 n \cdot \hat{a}n = \lambda. \tag{1.32}$$

It is well known (see [4],[3]) that if $a(y)$ is positive definite, then so is the homogenised matrix $\hat{a}$, therefore $n_i\hat{a}_{ij}n_j$ and λ are positive. Then we see that $k > 0$ and we have propagating waves for all frequencies.

Now let us consider the limit problem for (1.28)-(1.29) in the high contrast setting described in section 1.3. From the derivation of (1.24)-(1.20) we see that the form of the limit problem will not be altered if we change the function f. So upon replacing $f^\varepsilon(x)$ by λu^ε we arrive at the following homogenised problem:

$$-\mathrm{div}\left(A^{\mathrm{hom}}\nabla u_0\right) = \lambda\left(u_0(x) + \langle v(x,y)\rangle_{Q_0}\right), \quad \text{in } \Omega \tag{1.33}$$

$$u_0 = 0, \quad \text{on } \partial\Omega \tag{1.34}$$

$$-\Delta_y v(x,y) = \lambda\left(u_0(x) + v(x,y)\right) \quad y \in Q_0 \tag{1.35}$$

$$v(x,y) = 0, \qquad y \in \Gamma. \tag{1.36}$$

As we can see the system of equations (1.33)-(1.36) are coupled. Seeking a solution to (1.35)-(1.36) of the form $v(x,y) = \lambda u_0(x)V(y)$ the coupled system then becomes

$$-\mathrm{div}\left(A^{\mathrm{hom}}\nabla u_0\right) = \beta(\lambda)u_0(x), \quad \text{in } \Omega \tag{1.37}$$

$$u_0 = 0, \qquad \text{on } \partial\Omega \tag{1.38}$$

$$-\Delta_y V(x,y) - \lambda V(y) = 1, \qquad y \in Q_0 \tag{1.39}$$

$$V(y) = 0, \qquad y \in \Gamma \tag{1.40}$$

where $\beta(\lambda) = \lambda\left(1 + \lambda \langle V(y)\rangle_{Q_0}\right)$. To decouple this system we would have to find a solution for (1.39)-(1.40), if it exists, then evaluate $\beta(\lambda)$. Assuming that this is possible and continuing the analysis, seek a plane wave solution $u_0(x) = Ae^{(ikx\cdot n)}$ and substitute into (1.37)-(1.38) giving

$$\left[k^2 n \cdot \left(A^{\text{hom}} n\right) - \beta(\lambda)\right] A = 0. \tag{1.41}$$

As having an amplitude of zero will result in a trivial solution we then deduce that

$$k^2 n \cdot \left(A^{\text{hom}} n\right) - \beta(\lambda) = 0, \tag{1.42}$$

since A^{hom} is positive definite, then negative values of k^2 will occur if $\beta(\lambda) < 0$. This will result in exponentially decaying solutions. Therefore if there exists such values of λ that result in a negative valued β waves of that frequency cease to propagate.

1.5 Example of band gaps in isotropic high-contrast model

Let us now consider an example that will explicitly show band-gap effects, that is, show that there exist certain values of λ for which waves can not propagate. We shall consider the example $\Omega = \mathbb{R}^3$, $Q = [-1/2, 1/2]^3$ and the inclusions are balls in $\mathbb{R}^3$ with radius a, that is $Q_0 = \left\{x = (x_1, x_2, x_3) \in Q \mid x_1^2 + x_2^2 + x_3^2 < a^2\right\}$. Clearly Γ is the surface of the ball and let $a < 1/2$ so that the inclusions are isolated.

First concerning ourselves with (1.39)-(1.40), seeking a solution of the form $V = \hat{V} - \frac{1}{\lambda}$, gives

$$-\Delta_y \bar{V}(y) - \lambda \bar{V}(y) = 0, \qquad y \in \Omega_0 \tag{1.43}$$

$$\bar{V}(y) = \frac{1}{\lambda}, \quad y \in \Gamma \tag{1.44}$$

(1.43) admits a general solution of the form $\bar{V} = \frac{A}{|y|}\sin\left(\sqrt{\lambda}|y|\right)$, upon substitution into (1.44) we find $A = \frac{a}{\lambda}\text{cosec}\left(\sqrt{\lambda}a\right)$. So (1.39)-(1.40) admits the solution

$$V(y) = \frac{a}{\lambda}\text{cosec}\left(\sqrt{\lambda}a\right)\frac{\sin\left(\sqrt{\lambda}|y|\right)}{|y|} - \frac{1}{\lambda}. \tag{1.45}$$

From (1.45) and (1.7) we see that

$$\begin{aligned}
\langle V(y)\rangle_{Q_0} &= \int_{Q_0}\left(\frac{a}{\lambda}\text{cosec}\left(\sqrt{\lambda}a\right)\frac{\sin\left(\sqrt{\lambda}|y|\right)}{|y|} - \frac{1}{\lambda}\right)\text{dy} \\
&= \frac{a}{\lambda}\text{cosec}\left(\sqrt{\lambda}a\right)\underbrace{\int_{Q_0}\frac{\sin\left(\sqrt{\lambda}|y|\right)}{|y|}\text{dy}}_{\mathcal{I}} - \frac{1}{\lambda}|Q_0|
\end{aligned}$$

$\mathcal{I}$ is evaluated by transferring to spherical co-ordinates and noticing the function to be integrated is radially symmetric. Upon using integration by parts we arrive at

$$\mathcal{I} = 4\pi \left[\frac{1}{\lambda} \sin\left(\sqrt{\lambda} a\right) - \frac{1}{\sqrt{\lambda}} \left(a \cos\left(\sqrt{\lambda} a\right)\right)\right]$$

with $|Q_0| = \frac{4}{3}\pi a^3$, we arrive at

$$\langle V(y) \rangle_{Q_0} = \frac{4\pi a}{\lambda} \left[\frac{1}{\lambda} - \frac{1}{\sqrt{\lambda}} \left(a\text{cotan}\left(\sqrt{\lambda} a\right)\right) - \frac{a^2}{3}\right]$$

resulting in

$$\beta(\lambda) = \lambda \left(1 - \frac{4\pi a^3}{3} + \frac{4\pi a}{\lambda} \left[1 - \sqrt{\lambda} a\text{cotan}\left(\sqrt{\lambda} a\right)\right]\right)$$

as we can see β takes on negative values for an infinite number of values of λ.

Chapter 2

Rigorous formulation and appropriate analytic tools

As mentioned in the introduction, homogenisation theory is concerned with finding if a sequence of functions $\{u^\varepsilon\}$ 'converges' to some limit u^0, then finding the homogenised problem for which u^0 is a solution of. In order to attempt to address these problems we need to first define the space of functions in which our sequence lives.

2.1 Space of functions

Concerning ourselves again with the problem of a heterogeneous medium with periodic inclusions, as mentioned previously it is necessary to impose continuity of solution and flux across the boundary where the matrix and inclusion meet. Looking at the continuity of flux condition:

$$a(y)\tfrac{\partial u}{\partial x_i}n_i|_1 = a(y)\tfrac{\partial u}{\partial x_i}n_i|_0, \qquad y \in \Gamma \tag{2.1}$$

where n is the outward normal to the inclusion Q_0. We can see directly from (2.1) that ∇u is discontinuous. This means to justify the problem mathematically we have to take into account these discontinuities when defining the appropriate space in which the solution u exists.

Let us introduce the L^p spaces for $1 \leq p \leq \infty$. These are spaces of functions which are integrable with respect to the Lebesgue measure (see Section **??**), furthermore these spaces are normed vector spaces (Banach spaces) for their respective norms. More rigorously

Definition 2.1.1. If $1 \leq p < \infty$ then $L^p(\Omega)$ comprises of all Lebesgue measurable functions f on Ω for which $\int_\Omega |f|^p \mathrm{dx} < \infty$ and

$$\|f\|_p = \left(\int_\Omega |f|^p\right)^{1/p} \qquad \text{for } f \in L^p(\Omega) \tag{2.2}$$

$L^\infty(\Omega)$ comprises of all essentially bounded Lebesgue measurable functions on Ω and

$$\|f\|_\infty = \text{ ess sup}_\Omega |f| \quad \text{ for } f \in L^\infty(\Omega) \tag{2.3}$$

Taking into consideration the discontinuity of the gradient of u is done by considering a 'weak' derivative, which, is a variation of the derivative of 'sufficiently' smooth functions for those which are not so smooth. The definition of a weak derivative is as follows

Definition 2.1.2. Suppse $u, v \in L^1_{\text{loc}}(\Omega)$, we say that v is a weak derivative of u, written as

$$v = \frac{\partial u}{\partial x}$$

provided

$$\int_\Omega u \frac{\partial \varphi}{\partial x} \mathrm{dx} = -\int_\Omega v\varphi \mathrm{dx}\,, \qquad \forall \varphi \in C_0^\infty(\Omega) \tag{2.4}$$

Note that in general this definition of the weak derivative can be extended to include higher order derivatives (the α - weak derivative where α is a positive integer, see [5]. We now concern ourselves with Sobolev spaces, that are spaces of functions in which the function and its weak derivatives (up to a certain order depending on the Sobolev space) belong to L^p. Let us introduce the H^1 space given as

$$H^1(\Omega) := \left\{ u \mid u \in L^2(\Omega), \frac{\partial u}{\partial x_i} \in L^2(\Omega) \text{ for each } i = 1, 2, \ldots, n \right\} \tag{2.5}$$

which is a Hilbert space (a Banach space equipped with an inner product) with the following norm

$$\|u\|_{H^1(\Omega)} = \|u\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(\Omega)}. \tag{2.6}$$

2.2 Convergence

Upon defining the appropriate Sobolev space W for which the sequence of solutions $\{u^\varepsilon\}$ is a subset of, does this sequence convergence? In the case of homogenisation the questions that arise then are for a particular problem does the sequence of solutions $\{u^\varepsilon\} \subset W$ converge to a limit $u^0 \in W$ and if so how can we find this limit, i.e. what is the problem in which u^0 is the solution. There are various methods that address the problem of convergence, we shall outline how in the case of elliptic PDE's convergence is tested using Lax-Milgram's lemma.

Lax-Milgram's lemma states

Lemma 2.2.1. (**Lax-Milgram**) Let a be a continuous bi-linear form on a Hilbert space H and let $F \in H^*$ be a linear continuous functional. Assume that a is coercive on H. Then the problem (called the variational equation)

$$\begin{aligned} &\text{Find } u \in H \text{ such that} \\ &a(u, v) = \langle F, v \rangle\,, \quad \forall v \in H \end{aligned} \tag{2.7}$$

has a unique solution $u \in H$. Furthermore

$$\|u\|_H \leq C\|F\|_{H^*} \tag{2.8}$$

where C is a constant.

By definition a bilinear form is coercive on H if for any $v \in H$ there exists a $\nu > 0$ such that

$$a(v,v) \geq \nu \|v\|_H^2. \tag{2.9}$$

There is a well known result from functional analysis that states for a Hilbert space H, a bounded sequence $\{u^\varepsilon\} \subset H$ up to a subsequence weakly converges to a limit $u^0 \in H$. Then if it upon writing the homogenised problem in its variational form the assumptions of Lax-Milgram's lemma are satisfied, then via (2.8) we have a bounded sequence of solutions $\{u^\varepsilon\}$ and up to a subsequence weakly convergence to some limit u^0. The problem that remains then is to find if all of the sequence weakly converges to this limit u^0, which will be true if u^0 is independent of the subsequence, i.e. u^0 is a unique solution to some limit problem that is independent of ε. To find uniqueness of the limit we must first find the limit problem. If the limit problem is an elliptic problem then as mentioned above it is sufficient to apply Lax-Milgram lemma. Furthermore (in this case) bounds of the norm given by (2.8) can be used to approximate the original solution u^ε to the limit solution u^0. Note that finding the limit problem requires the use of other analytical tools such as, in the case of periodic homogenisation, compensated compactness and two-scale convergence.

2.3 Formal justification for limit problem of classical homogenisation problem

Here we shall consider the classical homogenisation problem for a heterogeneous periodic medium. Let us consider the following problem:

$$-\text{div}\left(a\left(\frac{x}{\varepsilon}\right)\nabla u^\varepsilon\right) = f^\varepsilon(x), \quad \text{in } \Omega \tag{2.10}$$

$$u^\varepsilon = 0, \qquad \text{on } \partial\Omega \tag{2.11}$$

where $f^\varepsilon(x) \in L^2[\Omega; H^1_\#(Q)]$. We shall consider an isotropic material, i.e. $a_{ij}(y) = a(y)\delta_{ij}$ where $a(y)$ is Q-periodic and for $0 < \alpha < \beta$

$$\alpha|\xi|^2 \leq a_{ij}\xi_i\xi_j \leq \beta|\xi|^2, \quad \forall \xi \in \mathbb{R}^n \tag{2.12}$$

Let us define

$$H^1_0(\Omega) := \left\{u \in H^1(\Omega) \text{ such that the trace of } u \text{ on } \Gamma \text{ equals zero}\right\}, \tag{2.13}$$

where from Friedrich's inequality

$$\|u\|_{H^1_0(\Omega)} \leq C\|\nabla u\|_{L^2(\Omega)}, \quad \forall u \in H^1_0(\Omega), C \text{ constant}$$

it can be shown that H^1_0 has the following equivalent norm

$$\|u\|_{H^1_0(\Omega)} = \int_\Omega |\nabla u|^2 \text{ dx}. \tag{2.14}$$

Multiplying (2.10) by a function $\varphi \in C^\infty_0(\Omega)$ and integrating by parts we arrive at

$$\int_\Omega a\left(\frac{x}{\varepsilon}\right)\nabla u^\varepsilon \nabla\varphi \text{ dx} = \int_\Omega f^\varepsilon\varphi \text{ dx}, \qquad \varphi \in C^\infty_0(\Omega) \tag{2.15}$$

which in light of Riesz Representation theorem and the fact that $L^2(\Omega) \subset H^{-1}(\Omega)$ $(H^{-1}(\Omega) = \left(H_0^1(\Omega)\right)^*)$ we can rewrite (2.10)-(2.11) in the following variational form

$$\begin{aligned} &\text{Find } u \in H_0^1(\Omega) \text{ such that} \\ &a(u,v) = \langle F, v \rangle, \quad \forall v \in H_0^1(\Omega) \end{aligned} \tag{2.16}$$

where

$$a(u,v) = \int_\Omega a\left(\frac{x}{\varepsilon}\right) \nabla u \nabla v \text{ dx}. \tag{2.17}$$

Equivalence of (2.10)-(2.11) and (2.16) can be seen by using our definition of a weak derivative (2.4) on (2.15). As we can see (2.15), known as the weak form of our problem, is more convenient as it contains all the information about the problem in one equation.

To find uniqueness of the solution for (2.16) we need to satisfy the assumption of the Lax-Milgram lemma, that is to show coercivity and continuity of the bilinear form (2.17). Coercivity can be easily shown from (2.12) as follows

$$\begin{aligned} a(v,v) &= \int_\Omega a\left(\frac{x}{\varepsilon}\right) \nabla v \nabla v \text{ dx} \\ &\geq \alpha \int_\Omega |\nabla v|^2 \text{ dx}. \end{aligned}$$

As saying a bilinear form is continuous is equivalent to saying the form is bounded, we can show that from conditions of $a(y)$ and Cauchy-Schwartz inequality that our bilinear form is bounded:

$$\begin{aligned} |a(u,v)| &= \left| \int_\Omega a\left(\frac{x}{\varepsilon}\right) \nabla u \nabla v \text{ dx} \right| \\ &\leq \beta \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)}. \end{aligned}$$

The above two calculations used the fact that (2.14) defines a norm on $H_0^1(\Omega)$. Therefore due to Lax-Milgram lemma u^ε is unique and bounded. Furthermore the sequence of solutions $\{u^\varepsilon\}$ corresponding to the problem (2.10)-(2.11) for different positive values of ε (which are tending to zero) is bounded and up to a subsequence weakly converge to some limit u^0.

As mentioned in Section 1.2 and can be shown ([4],[3]) by the use asymptotic expansions, our solution u^ε has the following form

$$u^\varepsilon(x,y) = u^{(0)}(x) + \varepsilon N_r(y) \frac{\partial u^{(0)}}{\partial x_r} + O(\varepsilon^2)$$

where $u^{(0)}(x)$ solves

$$-\operatorname{div}_x \left(\hat{a} \nabla u^{(0)}(x)\right) = F(x), \quad \text{in } \Omega \tag{2.18}$$

$$u^{(0)} = 0, \qquad \text{on } \partial\Omega \tag{2.19}$$

where $\hat{a} = \langle a(y) \left(\nabla_y N + I\right) \rangle$, $F(x) = \langle f(x,y) \rangle$ and $N(y)$ is the Y-periodic solution to the following problem:

$$-\operatorname{div}_y \left(a(y) \nabla_y N_j(y)\right) = \frac{\partial a}{\partial y_j}, \quad j = 1, 2, \ldots, n. \tag{2.20}$$

Mathematical justification for the form of the asymptotic expansion requires to show that solution to the auxiliary problem (2.20) exists. Multiplying (2.20) by a test function $\varphi \in C_0^\infty(Q)$ and integrating over Q we arrive the variational problem

$$\begin{aligned} &\text{Find } u \in H^1_\#(Q) \text{ such that} \\ &a(u,v) = \langle F, v \rangle\,, \quad \forall v \in H^1_\#(Q) \end{aligned} \tag{2.21}$$

where

$$a(u,v) = \int_Q a(y) \nabla u \nabla v \ \mathrm{dy} \tag{2.22}$$

Here we can not prove coercivity for the whole space $H^1_\#(Q)$ because (2.9) does not hold for the function $u =$ constant, which is indeed a function of the space $H^1_\#(Q)$. This can be overcome by the projection lemma (see [6]), that states for a Hilbert space H, let V be a (closed) subspace of H. Then every $x \in H$ can be uniquely written as $x = v + w$ where $z \in V$ and $w \in V^\perp$. Then if we let V be the space of constant functions from $H^1_\#(\Omega)$ then it is sufficient to prove coercivity for $V^\perp$, which can be easily seen from the definition of the $H^1_\#(Q)$ norm to be

$$V^\perp := \left\{ u \in H^1_\#(Q) \text{ such that } \langle u \rangle_Q = 0 \right\}$$

We can then see from Poincarè's inequality

$$\|u\|^2_{H^1}(\Omega) \leq C \left\{ \left(\int_\Omega u \mathrm{dx} \right)^2 + \int_\Omega |\nabla u|^2 \, \mathrm{dx} \right\}$$

that

$$\|u\|_{H^1_\#(Q)} \leq C \|\nabla u\|_{L^2(Q)}, \quad v \in V^\perp \tag{2.23}$$

and therefore

$$\begin{aligned} a(v,v) &= \int_\Omega a\left(\frac{x}{\varepsilon}\right) \nabla v \nabla v \ \mathrm{dx} \\ &\geq \alpha \int_\Omega |\nabla v|^2 \ \mathrm{dx} \\ &\geq \frac{\alpha}{C} \|v\|_{H^1_\#(Q)} \end{aligned}$$

which, as boundedness can easily been shown in the same way as before, Lax-Milgram's lemma gives us existence and uniqueness of $u \in V^\perp$. Extending this back to the space $H^1_\#(Q)$ we have existence and uniqueness upto a constant of the solution to the auxiliary problem (2.20).

It can also be shown that since a is positive definite, the homogenised matrix $\hat{a}$ is positive definite and since the homogenised problem is an elliptic boundary value problem, arguing the same way we did for (2.10)-(2.11) the solution u^0 to (2.18)-(2.19) exists and is unique. Therefore we know that $u^\varepsilon \rightharpoonup u^0$ in $H^1_0(\Omega)$. Furthermore it can be shown ([4],[3]) that

$$\|u^\varepsilon - u^{(0)}(x) - \varepsilon N_r \frac{\partial u^{(0)}}{\partial x_r} \|_{H^1(\Omega)} \leq C \varepsilon^{1/2},$$

where C a constant independent of ε.

2.4 Method of two-scale convergence

In this section we are going to introduce the definition of two-scale convergence and some of its properties (see [2] for proof of properties). Two scale convergence uses test functions which depend on both scales x and $\frac{x}{\varepsilon}$, the structure of these test functions make them the natural choice of being able to formally justify the two scale asymptotic expansion as we will shown soon in the case of a high contrasting medium. Another benefit of two-scale convergence is we obtain convergence without having to know the structure of the limit problem, where as in section 2 the convergence of the sequence of solutions $\{u^\varepsilon\}$ to the limit u^0 could only be shown after finding the homogenised equation $\hat{a}$ and showing it was elliptic via other means (see [3], [4] using auxiliary periodic problems). Let us introduce the definition of two-scale convergence

Definition 2.4.1. A sequence of functions u^ε, in $L^2(\Omega)$ is said to weakly two-scale converge to a limit $u^0(x,y) \in L^2(\Omega \times Q)$, if, for any function $\phi(x,y) \in \mathcal{D}[\Omega; C^\infty_\#(Q)]$, we have

$$\lim_{\varepsilon \to 0} \int_\Omega u^\varepsilon(x)\phi\left(x, \frac{x}{\varepsilon}\right) \mathrm{dx} = \int_\Omega \int_Q u^0(x,y)\phi(x,y)\mathrm{dxdy}$$

Notice that from definition 2.4.1 that two-scale convergence implies weak convergence. That is $u^\varepsilon \rightharpoonup u$ where $u = \frac{1}{|Q|}\int_Q u^0(x,y)$, if $u^0(x)$ then two-scale convergence coincides with weak convergence. Let us now introduce a few properties of two-scale convergence

Theorem 2.4.2. Let u^ε be a bounded sequence in $L^2(\Omega)$, where Ω is an open bounded set in $\mathbb{R}^n$. There exists a subsequence that two-scale converges to $u^0(x,y) \in L^2(\Omega \times Q)$.

Theorem 2.4.3. Strong two-scale convergence

Let u^ε be a sequence of functions in $L^2(\Omega)$ that two-scale converges to a limit $u^0(x,y) \in L^2(\Omega \times Q)$. Assume that

$$\lim_{\varepsilon \to 0} \|u^\varepsilon\|_{L^2(\Omega)} = \|u^0\|_{L^2(\Omega \times Q)}$$

Then, for any sequence v^ε that two-scale converges to a limit $v^0(x,y) \in L^2(\Omega \times Q)$, we have

$$u^\varepsilon(x)v^\varepsilon(x) \rightharpoonup \int_Q u^0(x,y)v^0(x,y)\mathrm{dy}$$

In section 1.3 we derived the homogenised problem for a high-contrast medium via asymptotic expansions. We shall now re derive these equations via the method of two-scale convergence. So we are concerning ourselves with a material made up of two components that are periodically distributed throughout the domain Ω with period εQ, where $Q = [0,1]^n$ is the unit cell. Letting component one (the inclusion) occupy Q_0 and component two (the matrix) occupy $Q_1 = Q \backslash Q_0$. Setting $\chi_i(y)$ for $i = 0,1$ to be characteristic function of Q_i extended by periodicity to $\mathbb{R}^n$, Ω is divided into two subdomains Ω_1^ε and Ω_0^ε:

$$\Omega_0^\varepsilon = \left\{x \in \Omega \mid \chi_0\left(\tfrac{x}{\varepsilon}\right) = 1\right\}, \quad \Omega_1^\varepsilon = \left\{x \in \Omega \mid \chi_1\left(\tfrac{x}{\varepsilon}\right) = 1\right\}$$

we shall assume that Ω_1^ε is a smooth connected domain. Let us consider the following problem

$$-\mathrm{div}\left(a\left(\frac{x}{\varepsilon}\right)\nabla u^\varepsilon\right) = f^\varepsilon(x), \quad \text{in } \Omega \tag{2.24}$$

$$u^\varepsilon = 0, \quad \text{on } \partial\Omega \tag{2.25}$$

where $f^{\varepsilon} \in L^2(\Omega; L^2_{\#}(Q))$ and bounded. Let

$$a(y) = \chi_1\left(\frac{x}{\varepsilon}\right) + \varepsilon^2\chi_0\left(\frac{x}{\varepsilon}\right) \tag{2.26}$$

Theorem 2.4.4. The sequence of solutions $\{u^{\varepsilon}\}$ to (2.24)-(2.25) two-scale converges to the limit $u^0(x,y) = u_0(x) + \chi_0(y)v(x,y)$, where (u_0, v) is the unique solution in $H^1_0(\Omega) \times L^2[\Omega; H^1_{0\#}(Q_0)]$ of the homogenised problem

$$-\text{div}\left(A^{\text{hom}}\nabla u_0\right) = < f(x,y) >_Q, \quad x \in \Omega \tag{2.27}$$

$$u_0 = 0, \qquad x \in \partial\Omega \tag{2.28}$$

$$-\Delta_y v(x,y) = f(x,y), \qquad y \in Q_0 \tag{2.29}$$

$$v(x,y) = 0, \qquad y \in \Gamma \tag{2.30}$$

where $A^{\text{hom}}_{ij} = \int_{Q_1}\left(\frac{\partial N_j}{\partial y_i} + \delta_{ij}\right)\text{dy}$.

We shall now state the following inequality without proof

$$\|w\|_{L^2(\Omega)} \leq C\left[\|\nabla w\|_{L^2(\Omega_1^{\varepsilon})} + \varepsilon\|\nabla w\|_{L^2(\Omega_0^{\varepsilon})}\right] \tag{2.31}$$

for any function $w \in H^1_0(\Omega)$, where C is a constant independent of ε. If we then multiply (2.24) by u^{ε}, integrate over Ω, then with the use of integration by parts and (2.25) we arrive at

$$\int_{\Omega} a(y)\,|\nabla u^{\varepsilon}|^2\,\text{dy} = \int_{\Omega} f^{\varepsilon}u^{\varepsilon}\text{dy} \tag{2.32}$$

Using Hölder's inequality on (2.32) we have

$$\left|\int_{\Omega} a(y)\,|\nabla u^{\varepsilon}|^2\,\text{dy}\right| = \left|\int_{\Omega} f^{\varepsilon}u^{\varepsilon}\text{dy}\right| \leq \|f\|_{L^2(\Omega)}\|u^{\varepsilon}\|_{L^2(\Omega)} = C\|u^{\varepsilon}\|_{L^2(\Omega)} \tag{2.33}$$

where C independent of ε. Then we see from (2.32)

$$\|\nabla u^{\varepsilon}\|^2_{L^2(\Omega_1^{\varepsilon})} \leq \|\nabla u^{\varepsilon}\|^2_{L^2(\Omega_1^{\varepsilon})} + \varepsilon^2\|\nabla u^{\varepsilon}\|^2_{L^2(\Omega_0^{\varepsilon})} \leq C\|u^{\varepsilon}\|_{L^2(\Omega)}$$

where last inequality is from (2.33). As we know that the problem (2.24)-(2.25) has a unique solution via Lax-Milgram lemma and we know that the solution is bounded we have the following estimate

$$\|\nabla u^{\varepsilon}\|_{L^2(\Omega_1^{\varepsilon})} \leq C \tag{2.34}$$

where C is independent of ε, note that C is a different constant from above but denoted by same symbol for brevity. A similar argument results in

$$\|\nabla u^{\varepsilon}\|_{L^2(\Omega_0^{\varepsilon})} \leq C \tag{2.35}$$

also note that (2.34),(2.35) and (2.31) give

$$\|u^{\varepsilon}\|_{L^2(\Omega)} \leq C \tag{2.36}$$

We know from Theorem 2.4.2 that bounded sequences have two-scale convergent subsequences. Therefore we know upto a subsequence $u^{\varepsilon}(x), \chi_1\left(\frac{x}{\varepsilon}\right)\nabla u^{\varepsilon}(x), \varepsilon\chi_0\left(\frac{x}{\varepsilon}\right)\nabla u^{\varepsilon}(x)$ two-scale converges to a limit $\eta_0(x,y), \eta_1(x,y), \eta_2(x,y)$ respectively. The limits can be worked out and the results and proof appear in [2], we shall just quote the results:

Lemma 2.4.5. There exists functions $u_0(x) \in H_0^1(\Omega), v(x,y) \in L^2[\Omega; H^1_{0\#}(Q_0)]$, and $u_1(x,y) \in L^2[\Omega; H^1_{\#}(Q_1)/\mathbb{R}]$ such that

$$\begin{aligned}
\eta_0(x,y) &= u_0(x) + \chi_0(y)v(x,y)\\
\eta_1(x,y) &= \chi_1(y)[\nabla u_0(x) + \nabla_y u_1(x,y)]\\
\eta_2(x,y) &= \chi_2(y)\nabla_y v(x,y)
\end{aligned}$$

Proof of Theorem 2.4.4 Let us choose a test function of the same form as the asymptotic expansion $\phi(x)+\varepsilon\phi_1\left(x,\frac{x}{\varepsilon}\right)+\Psi\left(x,\frac{x}{\varepsilon}\right)$, where $\phi(x)\in\mathcal{D}(\Omega), \phi_1(x,y)\in\mathcal{D}[\Omega; C^\infty_\#(Q)]$ and $\psi(x,y)\in\mathcal{D}[\Omega; C^\infty_\#(Q)]$ with $\psi(x,y)=0$ when $y\in Q_1$. The form of this test function is taken after seeing the structure of the limits given above. Multiplying (2.24) by the test function and integrating over the domain, we have

$$\int_\Omega -\mathrm{div}[a(y)\nabla u^\varepsilon]\left(\phi(x)+\varepsilon\phi_1\left(x,\frac{x}{\varepsilon}\right)+\chi_0\left(\frac{x}{\varepsilon}\right)\Psi\left(x,\frac{x}{\varepsilon}\right)\right)\mathrm{dx} =$$

$$\int_{\Omega_1^\varepsilon} -\mathrm{div}[\nabla u^\varepsilon]\left(\phi(x)+\varepsilon\phi_1\left(x,\frac{x}{\varepsilon}\right)\right)\mathrm{dx} + \int_{\Omega_0^\varepsilon} -\mathrm{div}[\varepsilon^2\nabla u^\varepsilon]\left(\phi(x)+\varepsilon\phi_1\left(x,\frac{x}{\varepsilon}\right)+\Psi\left(x,\frac{x}{\varepsilon}\right)\right)\mathrm{dx} =$$

$$\int_{\Omega_1^\varepsilon}\nabla u^\varepsilon\left(\frac{\partial\phi}{\partial x}+\frac{\partial\phi_1}{\partial y}\right)\mathrm{dx}+\varepsilon\left(\int_{\Omega_1^\varepsilon}\nabla u^\varepsilon\frac{\partial\phi_1}{\partial x}\mathrm{dx}+\int_{\Omega_0^\varepsilon}\nabla u^\varepsilon\frac{\partial\psi}{\partial y}\mathrm{dx}\right)+\varepsilon^2\left(\int_{\Omega_0^\varepsilon}\nabla u^\varepsilon\left[\frac{\partial\phi}{\partial x}+\frac{\partial\phi_1}{\partial y}+\frac{\partial\psi}{\partial x}\right]\right)\mathrm{dx}$$

$$+\varepsilon^3\int_{\Omega_0^\varepsilon}\nabla u^\varepsilon\frac{\partial\phi_1}{\partial x}\mathrm{dx} = \int_\Omega f^\varepsilon\left(\phi(x)+\varepsilon\phi_1\left(x,\frac{x}{\varepsilon}\right)+\chi_0\left(x,\frac{x}{\varepsilon}\right)\psi\left(x,\frac{x}{\varepsilon}\right)\right)\mathrm{dx}$$

Passing to the two-scale limit we have

$$\begin{aligned}\int_\Omega\int_{Q_1}[\nabla u_0(x)+\nabla_y u_1(x,y)]\left(\nabla\phi(x)+\nabla_y\phi_1(x,y)\right)\mathrm{dxdy}+\int_\Omega\int_{Q_0}\nabla_y v(x,y)\nabla_y\psi(x,y)\mathrm{dxdy}\\
=\int_\Omega\int_Q f(x,y)\left(\phi(x)+\chi_0(y)\psi(x,y)\right)\mathrm{dxdy}\end{aligned} \tag{2.37}$$

Notice that the two-scale limit of $f^\varepsilon(x)$ is $f(x,y)$ because the sequence $f^\varepsilon(x)$ is bounded and Q-periodic in y therefore upto a subsequence we have weak convergence in $L^2(Q)$ to its mean value $\langle f(x,y)\rangle_Q$ and therefore two-scale convergence is established. By density (2.37) holds for any $(\phi,\phi_1,\psi)\in H_0^1(\Omega)\times L^2[\Omega; H^1_\#(Q_1)/\mathbb{R}]\times L^2[\Omega; H^1_{0\#}(Q_0)]$. Where by definition this space has the norm

$$\|\nabla u_0(x)\|_{L^2(\Omega}+\|\nabla u_1(x,y)\|_{L^2(\Omega\times Q_1)}+\|\nabla v(x,y)\|_{L^2(\Omega\times Q_0)} \tag{2.38}$$

Therefore coercivity of $a(u,v)$ is easily established as $a(v,v)$ equals the norm given by (2.38). That is (2.37) admits a unique solution (u,u_1,v). Integrating (2.37) by parts one more time, while making the substitution $u^1(x,y)=N_j(y)\frac{\partial u_0}{\partial x_j}$, where $\{N\}_j=N_j(y)$ are Q periodic functions for $j,1,\ldots,n$, gives us the variational form of (2.24)-(2.25)

$$\begin{aligned}\int_\Omega\int_{Q_1}-\mathrm{div}\left(\left[I+\nabla_y N(y)\right]\nabla u_0(x)\right)\phi(x)\mathrm{dxdy}-\int_\Omega\int_{Q_0}\Delta_y v(x,y)\psi(x,y)\mathrm{dxdy}\\
=\int_\Omega\langle f(x,y)\rangle_Q\,\phi(x)\mathrm{dx}+\int_\Omega\int_{Q_0}f(x,y)\psi(x,y)\mathrm{dxdy}\end{aligned} \tag{2.39}$$

□

Chapter 3

Partially high contrasts in isolated elastic inclusions

In the recent paper [1] the wave propagation in periodic elastic composites with highly contrasting and highly anisotropic stiffnesses where considered. The results showed that for an elastic tensor of the form

$$C^{\varepsilon}(x) = \left\{ \begin{array}{rl} C^1, & x \in Q_1^{\varepsilon} \\ \varepsilon^2 C^0 + C^2, & x \in Q_0^{\varepsilon} \end{array} \right. ,$$

under the usual assumptions of symmetry and positive definiteness on C^0, C^1, with C^2 being symmetry and non-negative, band-gaps where present in the frequency spectrum of waves allowed to propagate. Furthermore directional propagation was shown, that is waves that would propagate in certain directions while cease to propagate in others. In the paper [1] a restriction was made on C^2 to find the homogenised problem, resulting in directional propagation being formally shown for a limited number of cases. In this Section we will show that for the case of isolated inclusions that the restriction can be removed from the problem formulation and via a different approach arrive at the same homogenised limit problem (given in [1]) and the same results mentioned thereafter. Then we shall consider a specific example, which does not satisfy the mentioned restriction, in an attempt to show the presence of band-gaps. For this example we shall in the same spirit as Section 2.2 try to justify the asymptotic expansion used.

3.1 Asymptotic expansions

Let us consider a "two-phase" heterogeneous periodic elastic material occupying Ω with highly contrasting and highly anisotropic constituents as described in section 1. Consider the wave equation

$$\rho_{\varepsilon} \ddot{u}^{\varepsilon}(x,t) - \operatorname{div} C^{\varepsilon} e^{\varepsilon} = f^{\varepsilon}(x,t) \tag{3.1}$$

with boundary conditions

$$u^{\varepsilon}|_1 = u^{\varepsilon}|_0, \quad \sigma_{ij}^{\varepsilon} n_j|_1 = \sigma_{ij}^{\varepsilon} n_j|_0 \tag{3.2}$$

which physically represent the continuity of the displacements and of the tractions across the interfaces respectively. Here n is the unit outward normal to Q_0^{ε}. The strain tensor σ is given by

$\sigma = Ce(u)$, where

$$e_{ij}(u) = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}$$

is the displacement vector. The density and the elasticity tensor are given by

$$\rho_\varepsilon(x) = \begin{cases} \rho_1, & x \in Q_1^\varepsilon \\ \rho_0, & x \in Q_0^\varepsilon \end{cases} \quad \text{and} \quad C^\varepsilon(x) = \begin{cases} C^1, & x \in Q_1^\varepsilon \\ \varepsilon^2 C^0 + C^2, & x \in Q_0^\varepsilon \end{cases}$$

where $C^0(y), C^1(y), C^2(y)$ are symmetric tensors, given by:

$$C_{ijpq} = C_{jipq} = C_{pqij} \tag{3.3}$$

and $C^r(y)$ for $r = 1, 2$ is strictly positive definite, given by:

$$\alpha_{ij} C^r_{ijpq} \alpha_{pq} > 0, \quad \text{for any symmetric tensor } \alpha, \alpha \neq 0 \tag{3.4}$$

while $C^2(y)$ is considered to be non-negative. Also the coefficients of the tensors $C^0(y), C^1(y), C^2(y)$ and $\rho^\varepsilon(y)$ are Q-periodic with period 1. Let $f^\varepsilon(x,t) = f^\varepsilon\left(x, \frac{x}{\varepsilon}, t\right)$ be a source term which may or may not locally vary. We are going to find the limit problem to (3.1)-(3.2) by the method of asymptotic expansion. That is we are going to consider the following formal expansion

$$u^\varepsilon(x,t) = u^{(0)}\left(x, \frac{x}{\varepsilon}, t\right) + \varepsilon u^{(1)}\left(x, \frac{x}{\varepsilon}, t\right) + \varepsilon^2 u^{(2)}\left(x, \frac{x}{\varepsilon}, t\right) + O(\varepsilon^3) \tag{3.5}$$

In Appendix A you can find the exact derivation of the results given in the remainder of this section. We see that from (5.10) $u^{(0)}(x,y,t) = u^0(x,t)$ in Q_1, which implies we look for a solution of the form

$$u^{(0)}(x,y,t) = \begin{cases} u^0(x,t), & x \in \Omega, \ y \in Q_1 \\ u^0(x,t) + v(x,y,t), & x \in \Omega, \ y \in Q_0 \end{cases} \tag{3.6}$$

where from simple substitution of the solution of this form into (5.6)-(5.8), together with the continuity condition $u^0|_1 = u^0|_0$, gives us the solvability condition for $v(x,y,t)$, namely:

$$-\frac{\partial}{\partial y_j}\left(C^2_{ijpq} \frac{\partial v_p}{\partial y_q}\right) = 0, \qquad y \in \Omega_0 \tag{3.7}$$

$$C^2_{ijpq} \frac{\partial v_p}{\partial y_q} n_j = 0, \qquad y \in \Gamma \tag{3.8}$$

$$v(x,y,t) = 0, \qquad y \in \Gamma \tag{3.9}$$

From (3.7) and (3.9), using integration by parts we see:

$$0 = -\int_{\Omega_0} \frac{\partial}{\partial y_j}\left(C^2_{ijpq} \frac{\partial v_p}{\partial y_q}\right) v_i = \int_{\Omega_0} \underbrace{C^2_{ijpq} \frac{\partial v_p}{\partial y_q} \frac{\partial v_i}{\partial y_j}}_{\geq 0}$$

which implies

$$C^2_{ijpq} \frac{\partial v_p}{\partial y_q} \equiv 0 \tag{3.10}$$

The function $v(x,y,t)$ which solves (3.7)-(3.9) is clearly not uniquely determined (if a solution exists at all), moreover we can see that the system also admits the trivial solution $v(x,y,t)=0$ which if was taken to be the solution to our problem would revert us back to the classical homogenisation case. Also note that if C^2 was indeed positive definite then the only solution that would be admitted is $v(x,y)\equiv 0$. Let us introduce from this point a space of functions which are admissible solutions to (3.7)-(3.9), that is

$$\mathcal{V} := \left\{ v \in H^1_{\#}(Q_0) \text{ such that } (3.7), (3.8) \ \& \ (3.9) \text{ hold.} \right\} \tag{3.11}$$

We also find that

$$u_p^{(1)}(x,y,t) = \begin{cases} N_s^{pr}(y) \frac{\partial u_r^0}{\partial x_s}(x,t), & y \in Q_1 \\ \\ N_s^{pr}(y) \frac{\partial u_r^0}{\partial x_s}(x,t) - y_r \frac{\partial v_p}{\partial x_r}(x,y,t), & y \in Q_0. \end{cases} \tag{3.12}$$

N is Q-periodic and is a solution of the problem (5.14)-(5.16), which expressed in it's weak form is

$$\int_Q C_{ijpq} \left(N_{s,q}^{pr} + \delta_{pr}\delta_{sq} \right) \frac{\partial \varphi}{\partial y_j} dy = 0, \qquad \forall \varphi \in \mathrm{C}^{\infty}_{\text{per}}(Q). \tag{3.13}$$

Our homogenised system of equations is

$$\langle \rho \rangle_Q \ddot{u}^0 + \langle \rho_0 \ddot{v} \rangle_{Q_0} - \operatorname{div}\left(C^{\text{hom}} \nabla u^0 \right) = \langle f \rangle_Q , \qquad x \in \Omega \tag{3.14}$$

$$\mathbf{P}\left[\rho_0 \left(\ddot{u}^0 + \ddot{v} \right) - \operatorname{div}_y \left(C^0 \nabla_y v \right) \right] = \mathbf{P}\left[f \right], \tag{3.15}$$

where C^{hom} is the homogenised matrix $C^{\text{hom}}_{ijpq} = \int_Q C_{ijpq} \left(\delta_{pr}\delta_{qs} + N^{rp}_{q,s} \right)$ in component form, δ is the Kronecker delta function, $\chi_0(y)$ is the characteristic function of Q_0 and $C_{ijpq} = \chi_0 C^2_{ijpq} + (1-\chi_0)\, C^1_{ijpq}$.

Where $\mathbf{P}$ is the orthogonal projection on the space $\mathcal{V}$, namely $\mathbf{P}f = \mathbf{P}g$ means

$$\int_{Q_0} f_i(y) w_i(y) \text{ dy} = \int_{Q_0} g_i(y) w_i(y) \text{ dy} \qquad \forall w \in \mathcal{V} \tag{3.16}$$

3.2 Weak formulation

As discussed in Section 2, the weak formulation is a convenient and powerful way of writing our governing equations. To this end we shall construct the weak form for our homogenised problem.

Let us multiply (3.14) by an arbitrary function $z(x,t) \in [C_0^{\infty}([0,\infty)); H_1^0(\Omega)]$ and integrate over our domain $\Omega \times [0,\infty)$

$$\int_0^{\infty} \int_{\Omega} \int_Q \rho \ddot{u}_i^0 z_i \text{ dxdydt} + \int_0^{\infty} \int_{\Omega} \int_{Q_0} \rho_0 \ddot{v}_i z_i \text{ dxdydt} - \int_0^{\infty} \int_{\Omega} \int_Q C_{ijpq} \left(\delta_{pr}\delta qs + N^{rp}_{q,s} \right) u^0_{p,jq} z_i \text{ dxdydt}$$
$$= \int_0^{\infty} \int_{\Omega} \int_Q f_i z_i \text{ dxdydt} \tag{3.17}$$

Taking (3.15), integrate over the domain, we get

$$
\begin{aligned}
-\int_0^\infty \int_\Omega \int_{Q_0} \rho_0 \left(\dot{u}_i^0 + \dot{v}_i\right) \dot{w}_i \ \mathrm{dxdydt} + \int_0^\infty \int_\Omega \int_{Q_0} C^0_{ijpq} \frac{\partial v_p}{\partial y_q} \frac{\partial w_i}{\partial y_j} \ \mathrm{dxdydt} \\
= \int_0^\infty \int_\Omega \int_{Q_0} f_i w_i \ \mathrm{dxdydt}
\end{aligned}
\tag{3.18}
$$

Extending $v, w \in \mathcal{V}$ by zero into Q_1 in (3.18) and combining (3.17) we arrive at

$$
\begin{aligned}
\int_0^\infty \int_\Omega C^{\mathrm{hom}}_{ijpq} u^0_{p,q} z_{i,j} \ \mathrm{dxdydt} + \int_0^\infty \int_\Omega \int_{Q_0} C^0_{ijpq} \frac{\partial v_p}{\partial y_q} \frac{\partial w_i}{\partial y_j} \ \mathrm{dxdydt} \\
- \int_0^\infty \int_\Omega \int_Q \rho \left(\dot{u}_i^0 + \dot{v}_i\right) \left(\dot{z}_i + \dot{w}_i\right) = \int_0^\infty \int_\Omega \int_Q f_i \left(z_i + w_i\right) \ \mathrm{dxdydt}
\end{aligned}
\tag{3.19}
$$

for any $(z, w) \in [C_0^\infty([0,\infty)); H_0^1(\Omega)] \times \mathcal{V} =: W$.

3.3 Example problem

Let us consider the following problem, let $f \equiv 0$, let our "two-scale" periodic composite elastic material with highly-contrasting and highly heterogeneous constituents occupy $\Omega \subset \mathbb{R}^3$. Let the inclusions be isolated spheres of radius a. Let the material within the inclusions be governed by the linear isotropic elastic tensor with Lame coefficients $\mu \sim \varepsilon^2$, $\lambda \sim 1$, such that our elasticity tensor within the inclusion is of the form

$$
C_{ijpq}(y) = \varepsilon^2 \left(\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jp}\right) + \delta_{ij}\delta_{pq}, \qquad y \in Q_0 \tag{3.20}
$$

The homogenised problem for the wave equation is then

$$
\langle \rho \rangle_Q \ddot{u}^0 + \langle \rho_0 \ddot{v} \rangle_{Q_0} - \operatorname{div}\left(C^{\mathrm{hom}} \nabla u^0\right) = 0, \qquad x \in \Omega \tag{3.21}
$$

$$
\mathbf{P}\left[\rho_0\left(\ddot{u}^0 + \ddot{v}\right) - \operatorname{div}_y\left(C^0 \nabla_y v\right)\right] = 0, \qquad y \in Q_0 \tag{3.22}
$$

We shall consider time harmonic solutions, that is $u^0(x,t) = e^{(i\omega t)}u^0(x), v(x,y,t) = e^{(i\omega t)}v(x,y)$, so (3.21)-(3.22) reduces to

$$
-\operatorname{div}\left(C^{\mathrm{hom}} \nabla u^0\right) = \Lambda\left(\langle \rho \rangle_Q u^0 + \langle \rho_0 v \rangle_{Q_0}\right), \qquad x \in \Omega \tag{3.23}
$$

$$
\mathbf{P}\left[-\operatorname{div}_y\left(C^0 \nabla_y v\right)\right] = \Lambda \rho_0 \mathbf{P}\left[\left(u^0 + v\right)\right], \qquad y \in Q_0 \tag{3.24}
$$

where $\Lambda = \omega^2$, $(u, v) \in W$. We will follow the same approach given in [1] and seek a solution to (3.21)-(3.22) of the form

$$
v_i(x,y) = \Lambda \eta_i^r(y) u_r^0(x). \tag{3.25}
$$

It can be easily seen that $\eta^r \in \mathcal{V}$. Upon substitution of (3.25) into (3.23) we have

$$
-\operatorname{div}\left(C^{\mathrm{hom}} \nabla u^0\right) = \beta(\Lambda) u^0, \qquad x \in \Omega \tag{3.26}
$$

$$
\tag{3.27}
$$

where

$$\beta_{ij}(\Lambda) = \Lambda \langle \rho \rangle_Q \delta_{ij} + \Lambda^2 \left\langle \rho_0 \eta_i^j \right\rangle_{Q_0}. \tag{3.28}$$

From substitution of (3.25) into (3.24) shows $\eta^r \in \mathcal{V}$ solves

$$\mathbf{P}\left[- \text{ div } \left(C^0 \nabla \eta^r\right) - \Lambda \rho_0 \eta^r\right] = \mathbf{P}\left[\rho_0 e^r\right], \quad y \in Q_0 \tag{3.29}$$

$$\eta^r = 0, \qquad y \in \Gamma. \tag{3.30}$$

where e^r is the unit co-ordinate vector. Now let us apply this to our specific example (3.20), we immediately see from (3.10) that $\mathcal{V}$ becomes the space of divergence free functions that disappear on the boundary Γ, that is

$$\mathcal{V} := \left\{v \in H^1_\#(Q) \mid \nabla_y \cdot v \equiv 0,\ y \in Q_0, v|_\Gamma = 0\right\} \tag{3.31}$$

and (3.29) reduces to

$$\mathbf{P}\left[\Delta \eta^r - \Lambda \rho_0 \eta^r\right] = \mathbf{P}\left[\rho_0 e^r\right], \quad y \in Q_0 \tag{3.32}$$

$$\tag{3.33}$$

from (3.16) the above equation can be rewritten as

$$\int_{Q_0} \eta^r_{i,jj} w_i - \Lambda \rho_0 \eta^r_i w_i \mathrm{dy} = \int_{Q_0} \rho \delta_{ri} w_i \mathrm{dy}, \quad \forall w \in \mathcal{V} \tag{3.34}$$

we shall recall the fact that the orthogonal of divergence free functions are the gradients (see [4]). Then from (3.34) we have

$$\Delta \eta^r - \Lambda \rho_0 \eta^r - \rho_0 e^r = \nabla p, \quad y \in Q_0.$$

A simple substitution $\eta^r = \eta^r - \frac{1}{\Lambda}$ above and we arrive at the following Stoke's problem

$$\Delta \eta^r - \Lambda \rho_0 \eta^r = \nabla p, \quad y \in Q_0 \tag{3.35}$$

$$\nabla \cdot \eta^r = 0, \qquad y \in Q_0 \tag{3.36}$$

$$\eta^r = \frac{1}{\Lambda} e^r, \quad y \in \Gamma. \tag{3.37}$$

As we can see (3.35) and (3.36) for a given $r = 1, 2, \ldots, 3$ is a system of four equations with four unknowns (η^r, p). This problem has yet to be solved.

3.4 Justification of solution to Unit Cell Problem

Although the asymptotic solution shows us that we can find a limit solution and a system of limit equations, it is only justified if indeed the unit cell problem (and then the homogenised problem) does have a solution. To this end is necessary to find existence and uniqueness of the solution to

$$\int_Q C_{ijpq} \left(N^{pr}_{s,q} + \delta_{pr}\delta_{sq}\right) \frac{\partial \varphi}{\partial y_j} dy = 0, \qquad \forall \varphi \in \mathrm{C}^\infty_{\mathrm{per}}(\Omega) \tag{3.38}$$

for the specific example of

$$C^{2}_{ijpq}(y) = \delta_{ij}\delta_{pq}, \qquad y \in Q_0 \tag{3.39}$$

We can see that

$$a(u,v) = \int_{Q_1} C^{1}_{ijpq} u_{p,q} v_{i,j} \ \mathrm{dy} + \int_{Q_0} (\nabla \cdot u)(\nabla \cdot v) \ \mathrm{dy}, \qquad u, v \in \left[H^1_{\#}(Q)\right]^n \tag{3.40}$$

Let us define an equivalent norm for $\left[H^1_{\#}(Q)\right]^n$ which we are going to use. That is

$$\|u\|_{H^1_{\#}(Q)} = \int_Q |\nabla u|^2 \ \mathrm{dy} + \left(\int_{Q_1} u \ \mathrm{dy}\right)^2 \tag{3.41}$$

To apply Lax-Milgram lemma it will be necessary to prove coercivity of (3.40). First notice that if $v \in V$ where

$$V := \left\{ v \in \left[H^1_{\#}(Q)\right]^n \ \mid v = \text{constant for } y \in Q_1, \nabla \cdot v = 0 \text{ for } y \in Q_0 \right\}$$

which we see that straight away for $v \in V$ then

$$a(v,v) = 0$$

and we therefore do not have coercivity of the bilinear form. We shall consider the space of functions that are orthogonal to V. By definition

$$V^{\perp} = \left\{ w \in \left[H^1_{\#}(Q)\right]^n \ \mid \ (w,v) = 0, \forall v \in V \right\}.$$

Let us now find explicitly the orthogonal space $V^{\perp}$. From (3.41) we have that the inner product of the Hilbert space $\left[H^1_{\#}(Q)\right]^n$ is

$$(w,v) = \int_Q (\nabla w)(\nabla v) \ \mathrm{dy} + \left(\int_{Q_1} w \ \mathrm{dy}\right)\left(\int_{Q_1} v \ \mathrm{dy}\right). \tag{3.42}$$

First of all let us consider the function $v = c$, where c is a constant, clearly $v \in V$ and then (3.42) reads

$$(w,v) = c\left(\int_{Q_1} w \ \mathrm{dy}\right), \qquad w \in \left[H^1_{\#}(Q)\right]^n \tag{3.43}$$

so for $(w,v) = 0$ then

$$\left(\int_{Q_1} w \ \mathrm{dy}\right) = 0, \qquad w \in \left[H^1_{\#}(Q)\right]^n. \tag{3.44}$$

Now let us consider all functions $v \in V$ which are equal to zero in Q_1. This will cover the remainder of the functions in V not equal to a constant because any function $v \in V$ can be written as $v = c + \bar{v}$, where c is a constant and

$$\bar{v} = \begin{cases} 0 & y \in Q_1 \\ \nabla \cdot v = 0 & y \in Q_0. \end{cases}$$

Therefore we have, via integration by parts

$$(w, v) = \int_{Q_0} (\nabla w) : (\nabla v) \ \mathrm{dy} = -\int_{Q_0} v \cdot \Delta w \ \mathrm{dy},$$

which leads to the necessary condition

$$\int_{Q_0} v \cdot \Delta w \ \mathrm{dy} = 0, \qquad v \in V, w \in \left[H^1_{\#}(Q)\right]^n. \tag{3.45}$$

It is known ([4]) that divergence free functions are orthogonal to gradients, which implies

$$\text{For } y \in Q_0 : \qquad \Delta w = \nabla \varphi, \qquad \text{where } \varphi \in L^2(Q_0). \tag{3.46}$$

So we have found explicitly the orthogonal space to V, i.e.

$$V^{\perp} := \left\{ w \in \left[H^1_{\#}(Q)\right]^n \text{ such that (3.44) and (3.46) holds.} \right\} \tag{3.47}$$

Conjecture:

For $w \in V^{\perp}$, the bilinear form (3.40) is coercive on $V^{\perp}$. That is there exists $\nu > 0$ such that

$$\int_{Q_1} \left(C^1 \nabla w\right) : \nabla w \ \mathrm{dy} + \int_{Q_0} (\nabla \cdot w)^2 \ \mathrm{dy} \geq \nu \int_{Q} |\nabla w|^2 \ \mathrm{dy}, \qquad \forall w \in V^{\perp}$$

Outline of Proof

Since we know that C^1 is symmetric, positive definite then for a given $\nu_1 > 0$

$$C^1 \nabla w : \nabla w = C^1 e(w) : e(w) \geq \nu_1 |e(w)|^2 \tag{3.48}$$

Therefore

$$\begin{aligned}
\int_{Q_1} \left(C^1 \nabla w\right) : \nabla w \ \mathrm{dy} + \int_{Q_0} (\nabla \cdot w)^2 \ \mathrm{dy} &\geq \nu_1 \int_{Q_1} |e(w)|^2 \ \mathrm{dy} + \int_{Q_0} (\nabla \cdot w)^2 \ \mathrm{dy} \\
&\geq \bar{\nu_1} \int_{Q_1} |\nabla w|^2 \ \mathrm{dy} + \int_{Q_0} (\nabla \cdot w)^2 \ \mathrm{dy} \\
&\geq \ \min \ \{\bar{\nu_1}, 1\} \left(\int_{Q_1} |\nabla w|^2 \ \mathrm{dy} + \int_{Q_0} (\nabla \cdot w)^2 \ \mathrm{dy} \right)
\end{aligned}$$

where the first inequality is give by (3.48) and the second inequality comes from Korn Inequality, which states that there exists a constant C such that

$$\int_{Q_1} |\nabla w|^2 \,\mathrm{dy} \leq C \int_{Q_1} |e(w)|^2 \,\mathrm{dy}, \quad \forall w \in V^{\perp}.$$

Therefore we see what we are left to prove is the following conjecture:

There exists $\nu_2 > 0$ such that

$$\int_{Q_1} |\nabla w|^2 \ \mathrm{dy} + \int_{Q_0} (\nabla \cdot w)^2 \ \mathrm{dy} \geq \nu_2 \int_{Q} |\nabla w|^2 \ \mathrm{dy}, \qquad \forall w \in V^{\perp} \tag{3.49}$$

This has not yet been proved. One possible direction is to take $0 < \nu_2 < 1$ then it would be sufficient to show (if possible) that there exists such a number ν_2 such that

$$\int_{Q_0} (\nabla \cdot w)^2 \ \mathrm{dy} \geq \nu_2 \int_{Q_0} |\nabla w|^2 \ \mathrm{dy}, \qquad \forall w \in V^{\perp}.$$

If $n = 1$ (one-dimensional case) then the above inequality is easily shown to be true as

$$(\nabla \cdot w)^2 = |\nabla w|^2 .$$

Chapter 4

Brief review of Lebesgue Integration

As we have seen in the previous sections, the results from functional analysis have been fundamental to homogenisation theory. To use this analysis it is necessary that the spaces we are working in are complete normed vector spaces (Banach spaces). In the theory of homogenisation we are concerned with spaces of integrable functions. It turns out that the set of all Riemann integrable functions is not complete which is why a more general form of integration is necessary that will accept more classes of functions and will form a complete set of integrable functions. This led to the notion of Lebesgue integration and the spaces of Lebesgue integrable functions (L^p spaces). In this section we are going to review the basic principles of Lebesgue integration and (without proofs) some of the properties of Lebesgue integrable functions. The material here was provided is taken from [7].

To find the area 'under the curve' of functions we will first need to define a concept of how to measure the area of geometric objects of any shape. One way of doing this is by comparing the area of the object to that of a shape with a known area for example a rectangle. For objects which can not be 'fitted' exactly with a finite number of rectangles, for example see figure 4, we can approximate the area of this object with a finite number of rectangles and in the limit of sending this number to infinity we would find the area of the object. To mathematically define this concept,

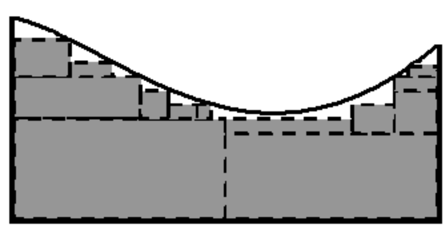

Figure 4.1: *Area of a geometric object being approximated by rectangles.*

we will introduce the definition of a σ-algebra:

Definition 4.0.1. A σ-algebra on a set X, is a family Σ of subsets of X satisfying:

1. $X \in \Sigma$

2. $S \in \Sigma \Rightarrow X \backslash S \in \Sigma$, (closed under complements)

3. $S_n \in \Sigma \forall n \in \mathbb{N} \Rightarrow \bigcup_{n=1}^{\infty} S_n \in \Sigma$, (closed under countable unions)

where (X, Σ) is called a measurable space. We also call the members of Σ measurable sets.

As we can see from the definition of a σ-algebra, if we have a geometric object (X) and we cut it up into an infinite number of arbitrarily small rectangles (S_n), then the set off all the rectangles and the 'left over bits' will form a Σ of X. All that is left to do then is to find a way of measuring the area of each individual element of Σ in such a way that adding all of these areas together would find the area of X. This idea leads to the following definition of a measure:

Definition 4.0.2. μ is called a measure on the measurable space (X, Σ) if

1. $\mu : \Sigma \mapsto [0, \infty]$

2. $\mu(\emptyset) = 0$

3. if $\{S_n\}_{n=1}^{\infty} \subset \Sigma$ are disjoint then $\mu\left(\bigcup_{n=1}^{\infty} S_n\right) = \sum_{n=1}^{\infty} \mu(S_n)$.

(X, Σ, μ) is called a measure space.

As we can see from our intuitive discussion above of how to measure the area of our object X, we would cut our shape into rectangles (possibly taking an infinite number of rectangles to do so) forming our Σ, then as we know the area of each rectangle exactly (property 1 of definition 4.0.2) and that adding all the squares together will reconstruct our shape X we use property 3 of definition 4.0.2 to find the area of our shape X.

As any function that satisfies definition 4.0.2 is a measure we are going to be concerned with the following measure, called the Lebesgue outer measure. First we define $\mathcal{P}(\mathbb{R}^n)$ as the set of all subsets of $\mathbb{R}^n$, also if $I = (a_1, b_1) \times \ldots \times (a_n, b_n)$ is an open rectangle in $\mathbb{R}^n$ the volume of I is given by $|I| = (b_1 - a_1)(b_2 - a_2) \ldots (b_n - a_n)$. Let us also say that the empty set $\emptyset$ is a rectangle of volume 0.

Definition 4.0.3. Let $S \in \mathcal{P}(\mathbb{R}^n)$. The Lebesgue outer measure of S is given by

$$m_0(S) = \inf \left\{ \sum_{n=1}^{\infty} |I_n| \,\middle|\, \{I_n\}_{n=1}^{\infty} \text{ a countable cover of} S \text{ by open rectangles} \right\}.$$

The first property to note is that if $I \subset \mathbb{R}^n$ then $m_0(I) = |I|$, so the Lebesgue outer measure of a rectangle is its area. Also note from the definition of the Lebesgue outer measure that it is not a measure on $\mathcal{P}(\mathbb{R}^n)$, so we need to restrict ourselves to a subset of $\mathbb{R}^n$ for which it is a measure.

Definition 4.0.4. A set $S \subset \mathbb{R}^n$ is called Lebesgue measurable if

$$m_0(U \cap S) + m_0(U \backslash S) = m_0(U), \quad \text{for all } U \in \mathcal{P}(\mathbb{R}^n)$$

The family of all Lebesgue measurable subsets of $\mathbb{R}^n$ is denoted by Σ_L. Σ_L is a σ-algebra of $\mathbb{R}^n$ and the restriction of m_0 to Σ_L is called the Lebesgue measure denoted by m, where m is a measure on $(\mathbb{R}^n, \Sigma_L)$. Now that we have mathematically defined a measure that describes the process of covering objects with squares to find it's area let us show how this is applied to finding integrals. Let us define a measurable function

Definition 4.0.5. Let (X, Σ) be a measurable space and let $f : X \mapsto \bar{\mathbb{R}}$. We say f is a (Σ) measurable function if

$$\{x \in X | \, f(x) > \alpha\} \in \Sigma \text{ for all } \alpha \in \mathbb{R}.$$

A Lebesgue measurable function on $\mathbb{R}^n$ is therefore a function that is measurable with respect to the measurable space $(\mathbb{R}^n, \Sigma_L)$. We shall see in a moment why this abstract definition of a measurable function is important. First let us introduce the notion of simple functions:

Definition 4.0.6. Let (X, Σ) be a measurable space. A non-negative simple function on X is a function $\varphi = \sum_{n=1}^{K} c_n \chi_{A_n}$ where $c_1, \ldots, c_K \in (0, \infty)$, χ_A is the characteristic function of A and $A_1, \ldots, A_k \in \Sigma$.

Now the abstract definition of measurable functions becomes clear with the following theorem (with proof given for visualisation).

Theorem 4.0.7. Measurable functions are monotone limits of simple functions

Let f be a non-negative measurable function on a measurable space (X, Σ). Then f is the point wise limit of an increasing sequence of simple functions on X.

Proof Let $n \in \mathbb{N}$ and write

$$\begin{aligned} A_j &= \left\{ x \in X | \, j2^{-n} \leq f(x) < (j+1)2^{-n} \right\} \text{ for } 0 \leq j < n2^n, \\ A_{n2^n} &= \left\{ x \in X | \, f(x) \geq n2^{-n} \right\} \end{aligned}$$

which are (from definition 4.0.5) measurable sets. Now (see figure 4) for $x \in X$, set

$$\varphi_n = \sum_{j=0}^{n2^n} j2^{-n} \chi_{A_j} = \max \, \left\{ \xi \in \{0, 2^{-n}, 2 \cdot 2^{-n}, 3 \cdot 2^{-n}, \ldots, n\} \middle| \, \xi \leq f(x) \right\}$$

Then each φ_n is a simple measurable function, $0 \leq \varphi_n \leq \varphi_{n+1} \leq f$, and $\varphi_n \to f$ as $n \to \infty$.

$\square$

Now it is easy to see what the integral of a simple function is, which is for a measure space (X, Σ, μ) we define

$$\int_X \varphi \mathrm{d}\mu = \sum_{n=1}^{K} c_n \mu(A_n).$$

Since we know that a non-negative measurable function can be represented by a monotone limit of simple functions, this leads to the following definition

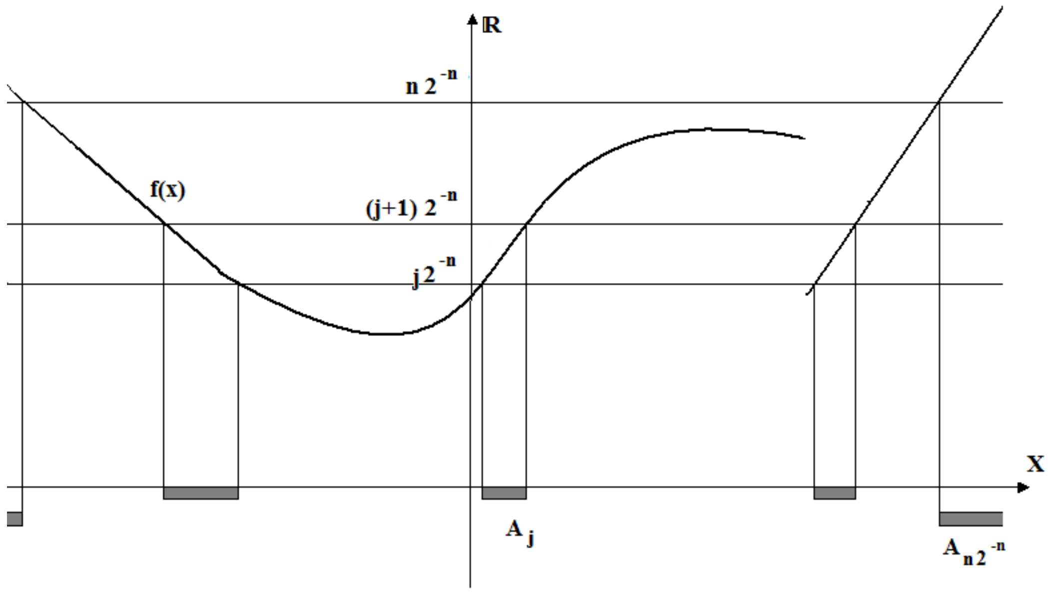

Figure 4.2: *Approximation of a function by simple functions.*

Definition 4.0.8. Integral of non-negative measurable functions Let (X, Σ, μ) be a measure space, $f : X \to [0, \infty]$ a non-negative measurable function. We define

$$\int_X f \mathrm{d}\mu = \sum \left\{ \int_X \varphi \mathrm{d}\mu \middle| \varphi \text{ simple}, 0 \leq \varphi \leq f \right\}.$$

So in analogy to finding the area of a geometric object. We have taken the area 'under the curve' as our object X, then cut it into horizontal strips of height ε for which the area is known then taken then area to be the sum of these horizontal strips in the limit of $\varepsilon \to 0$.

To extend this concept to all functions not just non-negative ones, we define the positive part and the negative part of a function $f : X \mapsto \bar{\mathbb{R}}$ to be $f_+(x) = \max\{f(x), 0\}$ and $f_-(x) = -\min(-f(x))_+$ for $x \in X$ respectively. Therefore $f = f_+ - f_-$ and $|f| = f_+ + f_-$, where is can be shown that if f is measurable on X then so is $f_+, f_- and |f|$. Where the integral of f is then given by

$$\int_X f \mathrm{d}\mu = \int_X f_+ \mathrm{d}\mu - \int_X f_- \mathrm{d}\mu.$$

We say that if both integrals on the right hand side are finite then f is integrable. From this definition we finally arrive at the definition of L^p spaces of Lebesgue measurable functions.

Definition 4.0.9. Let (X, Σ, μ) be a measure space. If $1 \leq p < \infty$ then $\mathcal{L}^p(X, \Sigma, \mu)$ comprises of all Σ-measurable functions f on X for which $\int_X |f|^p \mathrm{dx} < \infty$ and

$$\|f\|_p = \left(\int_X |f|^p \mathrm{d}\mu \right)^{1/p} \qquad \text{for } f \in L^p(X, \Sigma, \mu) \tag{4.1}$$

$L^\infty(X, \Sigma, \mu)$ comprises of all essentially bounded Σ-measurable functions on X and

$$\|f\|_\infty = \text{ ess sup}_X |f| \qquad \text{for } f \in L^\infty(X, \Sigma, \mu) \tag{4.2}$$

Although from the outset we can not define $\|f\|_p$ as a norm, we can do the following. Define $L^p(X, \Sigma, \mu)$ to be the set of equivalence classes of $\mathcal{L}^p(X, \Sigma, \mu)$ under the equivalence relation $f \sim g \Leftrightarrow f = g$ a.e. Define $[f] + [g] = [f + g]$, $\lambda[f] = [\lambda f]$, where $f, g \in \mathcal{L}^p$ are the finite-valued representatives of their equivalence classes, $\lambda \in \mathbb{R}$ and define $\|[f]\|_p = \|f\|_p$ for $f \in \mathcal{L}^p$. Then it can be shown that as $\|f\|_p = 0$ if and only if $f = 0$ a.e. then $[f] = 0$, and therefore $\|[f]\|_p$ defines a norm. Also it can be shown that $L^p(X, \Sigma, \mu)$ satisfies the axioms of a vector and therefore $L^p(X, \Sigma, \mu)$ is a real normed vector space. Finally it can be shown that $L^p(X, \Sigma, \mu)$ is complete (i.e. a Banach space). It is important to note that Riemann integrable functions as well as functions which are not Riemann integrable can be classified as Lebesgue integrable functions.

Chapter 5

Discussion

In this section we shall look at how the problem of in Section 3 could be developed further. In particular looking at the problem of propagation of elastic waves in the heterogeneous material with non-isolated inclusions.

When seeking an asymptotic solution of the form (3.5) to (3.1)-(3.2) then as shown we arrive at the following set of equations

$$-\frac{\partial}{\partial y_j}\left(C^1_{ijpq}\frac{\partial u_p^{(1)}}{\partial y_q}\right) = \frac{\partial}{\partial y_j}\left(C^1_{ijpq}u^0_{p,q}\right), \qquad y \in Q_1 \tag{5.1}$$

$$-\frac{\partial}{\partial y_j}\left(C^2_{ijpq}\frac{\partial u_p^{(1)}}{\partial y_q}\right) = \frac{\partial}{\partial y_j}\left(C^2_{ijpq}\left[\frac{\partial v_p}{\partial x_q} + u^0_{p,q}\right]\right), \qquad y \in Q_0. \tag{5.2}$$

These equations have to be satisfied for either non-isolated or isolated inclusions. This form of corrector problem has the additional coupling term

$$\frac{\partial}{\partial y_j}\left(C^2_{ijpq}\frac{\partial v_p}{\partial x_q}\right)$$

that is not present in the corrector problem's for the homogenisation problem (3.1)-(3.2) for moderate or highly contrasting coefficients of C^ε. This extra term was accounted for in the project with the introduction of solution of the form

$$u_p^{(1)}(x,y,t) = N_s^{pr}(y)\frac{\partial u_r^0(x,t)}{\partial x_s} + v^{(1)}, \quad y \in Q_0$$

where

$$v^{(1)} = -y_r\frac{\partial v_p}{\partial x_r}.$$

We can immediately see $v^{(1)}$ is not periodic so the solution $u^{(1)}$ is no longer periodic. Although this is overcome in the case of isolated inclusions by solving the problem on the reference cell Q then extending the solution by the periodicity of Q. This of course would not hold for non-isolated inclusions because of the intersection of the inclusions with the boundary of Q. Therefore to solve the problem in non-isolated inclusions requires the solution to (5.1)-(5.2) to be periodic. Seeking a solution of the form

$$u_p^{(1)}(x,y,t) = N_s^{pr}(y)\frac{\partial u_r^0(x,t)}{\partial x_s} + v^{(1)}(x,y,t)$$

where N_s^{pr} are the solutions of the linear elastic "unit cell" problems with periodic boundary conditions, which solves

$$\int_Q C_{ijpq}\left(N_{s,q}^{pr} + \delta_{pr}\delta_{sq}\right)\frac{\partial \varphi}{\partial y_j}dy = 0, \qquad \forall \varphi \in \mathrm{C}_{\mathrm{per}}^{\infty}(Q).$$

Similarly, $v_p^{(1)}$ are periodic solutions of the cell problem

$$\int_Q C_{ijpq}\left(\frac{\partial v_p^{(1)}}{\partial y_q} + \frac{\partial v_p}{\partial x_q}\right)\frac{\partial \varphi}{\partial y_j}dy = 0, \qquad \forall \varphi \in \mathrm{C}_{\mathrm{per}}^{\infty}(Q).$$

Here $C(y) := \chi_0(y)C^1 + (1-\chi_0(y))C^2$ with χ_0 being the characteristic function of Q_0. Without the restriction present in [1], for the case of non-isolated inclusions, it unclear on how to proceed. This is precisely because the solution $v^{(1)}$ is constrained by v, which 'strengthens' the coupling between their corresponding equations and the equations of solvability for $u^{(2)}$. This leads to an interesting open problem of formulating the homogenised problem for the case of non-isolated inclusions with the removal of the restriction presented in [1].

A final note on this problem is, that, although the wave equation was studied for an elastic material, it can be used to study a wider range of physical problems. One possible example is to use this problem in context of electromagnetism and study Maxwell's equations for the propagation of electromagnetic waves through an highly anisotropic two-phase heterogeneous material with the constituents having both moderate and highly contrasting physical properties.

Appendix A

Upon substituting (3.5) into (3.1), we arrive at the two following equations

$$\varepsilon^{-2}\left\{-\frac{\partial}{\partial y_j}\left(C^1_{ijpq}\frac{\partial u_p^{(0)}}{\partial y_q}\right)\right\}+\varepsilon^{-1}\left\{\frac{\partial}{\partial x_j}\left(C^1_{ijpq}\frac{\partial u_p^{(0)}}{\partial y_q}\right)+\frac{\partial}{\partial y_j}\left(C^1_{ijpq}\left[\frac{\partial u_p^{(0)}}{\partial x_q}+\frac{\partial u_p^{(1)}}{\partial y_q}\right]\right)\right\}$$
$$+\varepsilon^{0}\left\{\rho_\varepsilon\ddot{u}_i^{(0)}-\frac{\partial}{\partial y_j}\left(C^1_{ijpq}\left[\frac{\partial u_p^{(1)}}{\partial x_q}+\frac{\partial u_p^{(2)}}{\partial y_q}\right]\right)-\frac{\partial}{\partial x_j}\left(C^1_{ijpq}\left[\frac{\partial u_p^{(0)}}{\partial x_q}+\frac{\partial u_p^{(1)}}{\partial y_q}\right]\right)\right\}+O(\varepsilon)=f_i,\ y\in Q_1 \tag{5.3}$$

$$\varepsilon^{-2}\left\{-\frac{\partial}{\partial y_j}\left(C^2_{ijpq}\frac{\partial u_p^{(0)}}{\partial y_q}\right)\right\}+\varepsilon^{-1}\left\{\frac{\partial}{\partial x_j}\left(C^2_{ijpq}\frac{\partial u_p^{(0)}}{\partial y_q}\right)+\frac{\partial}{\partial y_j}\left(C^2_{ijpq}\left[\frac{\partial u_p^{(0)}}{\partial x_q}+\frac{\partial u_p^{(1)}}{\partial y_q}\right]\right)\right\}$$
$$+\varepsilon^{0}\left\{\rho_\varepsilon\ddot{u}_i^{(0)}-\frac{\partial}{\partial y_j}\left(C^2_{ijpq}\left[\frac{\partial u_p^{(1)}}{\partial x_q}+\frac{\partial u_p^{(2)}}{\partial y_q}\right]\right)-\frac{\partial}{\partial x_j}\left(C^2_{ijpq}\left[\frac{\partial u_p^{(0)}}{\partial x_q}+\frac{\partial u_p^{(1)}}{\partial y_q}\right]\right)-\frac{\partial}{\partial y_j}\left(C^0_{ijpq}\frac{\partial u_p^{(0)}}{\partial y_q}\right)\right\}$$
$$+O(\varepsilon)=f_i,\qquad y\in Q_0 \tag{5.4}$$

and substituting (3.5) into (3.2) gives:

$$\varepsilon^{-1}\left\{C^1_{ijpq}\frac{\partial u_p^{(0)}}{\partial y_q}\bigg|_1 n_j-C^2_{ijpq}\frac{\partial u_p^{(0)}}{\partial y_q}\bigg|_0 n_j\right\}+\varepsilon^{0}\left\{C^1_{ijpq}\left(\frac{\partial u_p^{(0)}}{\partial x_q}+\frac{\partial u_p^{(1)}}{\partial y_q}\right)\bigg|_1 n_j-C^2_{ijpq}\left(\frac{\partial u_p^{(0)}}{\partial x_q}+\frac{\partial u_p^{(1)}}{\partial y_q}\right)\bigg|_0 n_j\right\}$$
$$+\varepsilon^{1}\left\{C^1_{ijpq}\left(\frac{\partial u_p^{(2)}}{\partial y_q}+\frac{\partial u_p^{(1)}}{\partial x_q}\right)\bigg|_1 n_j-C^2_{ijpq}\left(\frac{\partial u_p^{(2)}}{\partial y_q}+\frac{\partial u_p^{(1)}}{\partial x_q}\right)\bigg|_0 n_j-C^0_{ijpq}\frac{\partial u_p^{(0)}}{\partial y_q}\bigg|_0 n_j\right\}+O(\varepsilon^2)=0 \tag{5.5}$$

Equating powers of order ε^{-2} in (5.3) and (5.4) gives

$$-\frac{\partial}{\partial y_j}\left(C^1_{ijpq}\frac{\partial u_p^{(0)}}{\partial y_q}\right)=0,\qquad y\in Q_1 \tag{5.6}$$

$$-\frac{\partial}{\partial y_j}\left(C^2_{ijpq}\frac{\partial u_p^{(0)}}{\partial y_q}\right)=0,\qquad y\in Q_0 \tag{5.7}$$

and equating powers of order ε^{-1} in (5.5):

$$C^1_{ijpq}\frac{\partial u_p^{(0)}}{\partial y_q}\bigg|_1 n_j=C^2_{ijpq}\frac{\partial u_p^{(0)}}{\partial y_q}\bigg|_0 n_j \tag{5.8}$$

Notice that from multiplying (5.6),(5.7) by $u_i^{(0)}$ and integrating over their respective domains we have :

$$\int_{Q_1} C^1_{ijpq}\frac{\partial u_p^{(0)}}{\partial y_q}\frac{\partial u_i^{(0)}}{\partial y_j}\mathrm{dy} + \int_{Q_0} C^2_{ijpq}\frac{\partial u_p^{(0)}}{\partial y_q}\frac{\partial u_i^{(0)}}{\partial y_j}\mathrm{dy} = -\int_{\Gamma}\left(C^1_{ijpq}\frac{\partial u_p^{(0)}}{\partial y_q}\bigg|_1 n_j - C^2_{ijpq}\frac{\partial u_p^{(0)}}{\partial y_q}\bigg|_0 n_j\right)\mathrm{dS}, \tag{5.9}$$

using the fact that if f is Q-periodic then $\int_{\partial Q} f(y)\mathrm{dy} = 0$, where ∂Q is the surface of Q. Notice that from (5.8) the right hand side of (5.9) equals zero. Now using the positive definiteness of C_1 and the fact C_2 is non-negative we have

$$\underbrace{\int_{Q_1} C^1_{ijpq}\frac{\partial u_p^{(0)}}{\partial y_q}\frac{\partial u_i^{(0)}}{\partial y_j}}_{>\nu\left|\nabla_y(u^{(0)})\right|^2} + \underbrace{\int_{Q_0} C^2_{ijpq}\frac{\partial u_p^{(0)}}{\partial y_q}\frac{\partial u_i^{(0)}}{\partial y_j}}_{\geq 0} = 0,$$

which implies

$$\left|\nabla_y(u^{(0)})\right| = 0, \quad y \in Q_1. \tag{5.10}$$

Equating powers of ε^{-1} in (5.3), (5.4), using (3.6) and (3.10) gives:

$$-\frac{\partial}{\partial y_j}\left(C^1_{ijpq}\frac{\partial u_p^{(1)}}{\partial y_q}\right) = \frac{\partial}{\partial y_j}\left(C^1_{ijpq}u^0_{p,q}\right), \qquad y \in Q_1 \tag{5.11}$$

$$-\frac{\partial}{\partial y_j}\left(C^2_{ijpq}\frac{\partial u_p^{(1)}}{\partial y_q}\right) = \frac{\partial}{\partial y_j}\left(C^2_{ijpq}\left[\frac{\partial v_p}{\partial x_q} + u^0_{p,q}\right]\right), \qquad y \in Q_0 \tag{5.12}$$

Seek a solution of the form

$$u_p^{(1)}(x,y,t) = \begin{cases} N_s^{pr}(y)\frac{\partial u_r^0}{\partial x_s}(x,t), & y \in Q_1 \\ \\ N_s^{pr}(y)\frac{\partial u_r^0}{\partial x_s}(x,t) - y_r\frac{\partial v_p}{\partial x_r}(x,y,t), & y \in Q_0 \end{cases} \tag{5.13}$$

therefore (5.11) and (5.12) become

$$\frac{\partial}{\partial y_j}\left(C^1_{ijpq}\frac{\partial N_s^{pr}}{\partial y_q}\right) = -\frac{\partial}{\partial y_j}\left(C^1_{ijrs}\right), \qquad y \in Q_1 \tag{5.14}$$

$$\frac{\partial}{\partial y_j}\left(C^2_{ijpq}\frac{\partial N_s^{pr}}{\partial y_q}\right) = -\frac{\partial}{\partial y_j}\left(C^2_{ijrs}\right), \qquad y \in Q_0. \tag{5.15}$$

Equating the powers of ε^0 in the boundary condition (5.5) gives

$$C^1_{ijpq}\left(u^0_{p,q} + N^{pr}_{s,q}u^0_{r,s}\right)n_j = C^2_{ijpq}\left(u^0_{p,q} + N^{pr}_{s,q}u^0_{r,s}\right)n_j \tag{5.16}$$

Equating powers of ε^0 in (5.3) and (5.4) we have

$$\frac{\partial}{\partial y_j}\left(C^1_{ijpq}\frac{\partial u_p^{(2)}}{\partial y_q}\right) = \rho_1\ddot{u}_i^0 - f_i - \frac{\partial}{\partial y_j}\left(C^1_{ijpq}\right)\frac{\partial u_p^{(1)}}{\partial x_q} - C^1_{ijpq}u^0_{p,qj} - C^1_{ijpq}\left(N^{pr}_{s,q}u^0_{r,sj} + N^{pr}_{s,j}u^0_{r,sq}\right), \qquad y \in Q_1 \tag{5.17}$$

$$\frac{\partial}{\partial y_j}\left(C^2_{ijpq}\frac{\partial u^{(2)}_p}{\partial y_q}\right) = \rho_0\left(\ddot{u}^0_i + \ddot{v}_i\right) - f_i - C^2_{ijpq}u^0_{p,jq} - \frac{\partial}{\partial y_j}\left(C^0_{ijpq}\frac{\partial v_p}{\partial y_q}\right) - C^2_{ijpq}\left[N^{pr}_{s,j}u^0_{r,sq} + N^{pr}_{s,q}u^0_{r,sj}\right]$$
$$+ C^2_{ijpq}y_r\frac{\partial^3 v_p}{\partial x_r\partial x_q\partial y_j} + c^2_{ijpq}\frac{\partial^2 v_p}{\partial x_j\partial x_q} - \frac{\partial}{\partial y_j}\left(C^2_{ijpq}\right)\left[N^{pr}_s u^0_{r,sq} - y_r\frac{\partial^2 v_p}{\partial x_r\partial x_q}\right], \qquad y\in Q_0 \tag{5.18}$$

and equating the power of ε^1 in (5.5) gives

$$C^1_{ijpq}\frac{\partial u^{(2)}_p}{\partial y_q}\bigg|_1 n_j - C^2_{ijpq}\frac{\partial u^{(2)}_p}{\partial y_q}\bigg|_0 n_j = -C^1_{ijpq}N^{pr}_s u^0_{r,sq}n_j + C^0_{ijpq}\frac{\partial v_p}{\partial y_q}n_j + C^2_{ijpq}\left(N^{pr}_s u^0_{r,sq} - y_r\frac{\partial^2 v_p}{\partial x_r\partial x_q}\right)n_j. \tag{5.19}$$

First solvability condtion for $u^{(2)}$.

From greens formula and remembering the normal is unit outward normal to Q_0 we have:

$$\int_{Q_0}\frac{\partial}{\partial y_j}\left(C^2_{ijpq}\frac{\partial u^{(2)}_p}{\partial y_q}\right)\mathrm{dy} + \int_{Q_1}\frac{\partial}{\partial y_j}\left(C^1_{ijpq}\frac{\partial u^{(2)}_p}{\partial y_q}\right)\mathrm{dy} = -\int_\Gamma\left(C^1_{ijpq}\frac{\partial u^{(2)}_p}{\partial y_q}\bigg|_1 n_j - C^2_{ijpq}\frac{\partial u^{(2)}_p}{\partial y_q}\bigg|_0 n_j\right)\mathrm{dS}. \tag{5.20}$$

Substituting (5.17)-(5.19) into (5.20) we have:

$$\int_Q\left(\rho\ddot{u}^0_i - f_i\right)\mathrm{dy} + \int_{Q_0}\rho_0\ddot{v}^0_i\,\mathrm{dy} - \int_Q\left(C_{ijpq}u^0_{p,jq} + C_{ijpq}\left[N^{pr}_{s,q}u^0_{r,sj} + N^{pr}_{s,j}u^0_{r,sq}\right] + \frac{\partial}{\partial y_j}\left(C_{ijpq}\right)N^{pr}_s u^0_{r,sq}\right)\mathrm{dy}$$
$$+ \int_{Q_0}\left\{-\frac{\partial}{\partial y_j}\left(C^0_{ijpq}\frac{\partial v_p}{\partial y_q}\right) + C^2_{ijpq}y_r\frac{\partial^3 v_p}{\partial x_r\partial x_q\partial y_j} + C^2_{ijpq}\frac{\partial^2 v_p}{\partial x_j\partial x_q} + \frac{\partial}{\partial y_j}\left(C^2_{ijpq}\right)y_r\frac{\partial^2 v_p}{\partial x_r\partial x_q}\right\}\mathrm{dy} =$$
$$\int_\Gamma\left(-C^0_{ijpq}\frac{\partial v_p}{\partial y_q}n_j - C^2_{ijpq}N^{pr}_s u^0_{r,sq}n_j + C^1_{ijpq}N^{pr}_s u^0_{r,sq}n_j + C^2_{ijpq}y_r\frac{\partial^2 v_p}{\partial x_r\partial x_q}n_j\right)\mathrm{dS}. \tag{5.21}$$

We can see from integration by parts that

$$-\int_Q C_{ijpq}N^{pr}_{s,j}u^0_{r,sq}\,\mathrm{dy} = \int_Q\frac{\partial}{\partial y_j}\left(C_{ijpq}\right)N^{pr}_s u^0_{r,sq}\,\mathrm{dy} + \int_\Gamma\left(C^1_{ijpq}N^{pr}_s u^0_{r,sq}n_j - C^2_{ijpq}N^{pr}_s u^0_{r,sq}n_j\right)\mathrm{dS},$$

and from divergence theorem

$$-\int_{Q_0}\frac{\partial}{\partial y_j}\left(C^0_{ijpq}\frac{\partial v_p}{\partial y_q}\right)\mathrm{dy} = -\int_\Gamma C^0_{ijpq}\frac{\partial v_p}{\partial y_q}n_j\,\mathrm{dS},$$

finally notice that

$$\int_{Q_0}C^2_{ijpq}y_r\frac{\partial^3 v_p}{\partial x_r\partial x_q\partial y_j}\mathrm{dy} = -\int_{Q_0}\left\{\frac{\partial}{\partial y_j}\left(C^2_{ijpq}\right)y_r\frac{\partial^2 v_p}{\partial x_r\partial x_q} + C^2_{ijpq}\frac{\partial^2 v_p}{\partial x_j\partial x_q}\right\}\mathrm{dy} + \int_\Gamma C^2_{ijpq}y_r\frac{\partial^2 v_p}{\partial x_r\partial x_q}n_j\,\mathrm{dS}.$$

Therefore (5.21) reduces to

$$\int_Q\rho\ddot{u}^0_i\,\mathrm{dy} + \int_{Q_0}\rho_0\ddot{v}_i\,\mathrm{dy} - \int_Q C_{ijpq}\left(\delta_{pr}\delta_{qs} + N^{rp}_{q,s}\right)u^0_{p,jq}\,\mathrm{dy} = \int_Q f_i\,\mathrm{dy}. \tag{5.22}$$

Second solvability condition:

Multiplying (5.18) by any $w \in \mathcal{V}$ and integrating over Q_0:

$$\int_{Q_0} \frac{\partial}{\partial y_j}\left(C^2_{ijpq}\frac{\partial u_p^{(2)}}{\partial y_q}\right) w_i \text{ dy} = \int_{Q_0} \left\{\rho_0\left(\ddot{u}_i^0 + \ddot{v}_i\right) - f_i\right\} w_i \text{ dy} - \int_{Q_0} C^2_{ijpq} u^0_{p,jq} w_i \text{ dy}$$
$$- \int_{Q_0} \frac{\partial}{\partial y_j}\left(C^0_{ijpq}\frac{\partial v_p}{\partial y_q}\right) w_i \text{ dy} - \int_{Q_0} C^2_{ijpq}\left[N^{pr}_{s,j} u^0_{r,sq} + N^{pr}_{s,q} u^0_{r,sj}\right] w_i \text{ dy}$$
$$+ \int_{Q_0} C^2_{ijpq} y_r \frac{\partial^3 v_p}{\partial x_r \partial x_q \partial y_j} w_i \text{ dy} + \int_{Q_0} C^2_{ijpq}\frac{\partial^2 v_p}{\partial x_j \partial x_q} w_i \text{ dy} - \int_{Q_0} \frac{\partial}{\partial y_j}\left(C^2_{ijpq}\right)\left[N^{pr}_s u^0_{r,sq} - y_r\frac{\partial^2 v_p}{\partial x_r \partial x_q}\right] w_i \text{ dy}. \tag{5.23}$$

Notice that

$$\int_{Q_0} \frac{\partial}{\partial y_j}\left(C^2_{ijpq}\frac{\partial u_p^{(2)}}{\partial y_q}\right) w_i \text{ dy} = -\int_{Q_0} C^2_{ijpq}\frac{\partial u_p^{(2)}}{\partial y_q}\frac{\partial w_i}{\partial y_j} \text{ dy} = 0, \tag{5.24}$$

$$-\int_{Q_0} C^2_{ijpq} N^{pr}_{s,j} u^0_{r,sq} w_i \text{ dy} = \int_{Q_0} \underbrace{C^2_{ijpq}\frac{\partial w_i}{\partial y_j}}_{=0} N^{pr}_s u^0_{r,sj} \text{ dy} + \int_{Q_0} \frac{\partial}{\partial y_j}\left(C^2_{ijpq}\right) w_i N^{pr}_s u^0_{r,sq} \text{ dy}, \tag{5.25}$$

and

$$\int_{Q_0} C^2_{ijpq} y_r \frac{\partial^3 v_p}{\partial x_r \partial x_q \partial y_j} w_i \text{ dy} =$$
$$-\int_{Q_0} \underbrace{C^2_{ijpq}\frac{\partial w_i}{\partial y_j}}_{=0} y_r \frac{\partial^2 v_p}{\partial x_r \partial x_q} \text{ dy} - \int_{Q_0} C^2_{ijpq}\frac{\partial^2 v_p}{\partial x_j \partial x_q} w_i \text{ dy} - \int_{Q_0} \frac{\partial}{\partial y_j}\left(C^2_{ijpq}\right) y_r \frac{\partial^2 v_p}{\partial x_r \partial x_q} w_i \text{ dy}. \tag{5.26}$$

Therefore (5.23) reduces to

$$\int_{Q_0} \rho_0\left(\ddot{u}_i^0 + \ddot{v}_i\right) w_i \text{ dy} - \int_{Q_0} \frac{\partial}{\partial y_j}\left(C^0_{ijpq}\frac{\partial v_p}{\partial y_q}\right) w_i \text{ dy} - \int_{Q_0} C^2_{ijpq} u^0_{p,jq} w_i \text{ dy} - \int_{Q_0} C^2_{ijpq} N^{pr}_{s,q} u^0_{r,sj} w_i \text{ dy}$$
$$= \int_{Q_0} f_i w_i \text{ dy}. \tag{5.27}$$

(5.27) can be further reduced by the use of the following result

Conjecture: $\Phi(x) \equiv 0$, where

$$\Phi(x) = \int_{Q_0} C^2_{ijrs}\left(N^{rp}_{p,s} + \delta_{pr}\delta_{qs}\right) u^0_{p,qj} w_i dy, \qquad w_i \in \mathrm{V}$$

Proof

By multiply $\Phi(x)$ by an arbitrary test function $\phi(x)$ and integrate over Ω and using integration by parts:

$$
\begin{aligned}
\int_\Omega \int_{Q_0} \phi(x) C^2_{ijrs} \left(N^{rp}_{q,s} + \delta_{pr}\delta_{qs}\right) u^0_{p,qj} w_i \ \mathrm{dxdy} &= \int_\Omega \int_{Q_0} \phi(x) C^2_{ijrs} \left(N^{rp}_{q,s} + \delta_{pr}\delta_{qs}\right) u^0_p \frac{\partial^2 w_i}{\partial x_q \partial x_j} \ \mathrm{dxdy} \\
= \ \int_\Omega \int_{Q_0} \phi(x) \frac{\partial}{\partial y_j} \left(y_r \frac{\partial^2 w_i}{\partial x_q \partial x_r}\right) C^2_{ijrs} \left(N^{rp}_{q,s} + \delta_{pr}\delta_{qs}\right) u^0_p \ \mathrm{dxdy} \\
= \ -\int_\Omega \int_{Q_0} \phi(x) y_r \frac{\partial^2 w_i}{\partial x_q \partial x_r} u^0_p \left[\frac{\partial}{\partial y_j} \left(C^2_{ijrs} \left(N^{rp}_{q,s} + \delta_{pr}\delta_{qs}\right)\right)\right] \ \mathrm{dxdy}, \quad \forall \phi(x).
\end{aligned}
\tag{5.28}
$$

as we can see from (5.15) the right hand side of the above (5.28) equals zero and therefore this implies

$$
\int_{Q_0} C^2_{ijrs} \left(N^{rp}_{p,s} + \delta_{pr}\delta_{qs}\right) u^0_{p,qj} w_i \ \mathrm{dy} \equiv 0. \tag{5.29}
$$

$\square$

Therefore (5.27) reduces to

$$
\int_{Q_0} \rho_0 \left(\ddot{u}^0_i + \ddot{v}_i\right) w_i \ \mathrm{dy} - \int_{Q_0} \frac{\partial}{\partial y_j} \left(C^0_{ijpq} \frac{\partial v_p}{\partial y_q}\right) w_i \ \mathrm{dy} = \int_{Q_0} f_i w_i \ \mathrm{dy}. \tag{5.30}
$$

(5.22) and (5.30) is our homogenised system of equations:

$$
\langle \rho \rangle_Q \ddot{u}^0 + \langle \rho_0 \ddot{v} \rangle_{Q_0} - \operatorname{div}\left(C^{\text{hom}} \nabla u^0\right) = \langle f \rangle_Q , \qquad x \in \Omega \tag{5.31}
$$

$$
\mathbf{P}\left[\rho_0 \left(\ddot{u}^0 + \ddot{v}\right) - \operatorname{div}_y \left(C^0 \nabla_y v\right)\right] = \mathbf{P}\left[f\right] \quad y \in Q_0, \ x \in Q. \tag{5.32}
$$

Bibliography

[1] V.P.Smyshlyaev, *Propagation and localisation of elastic waves in highly anisotropic periodic composite via homogenisation*, Mechanics of Materials, in press (2008).

[2] G.Allaire, *Homogenization and two-scale convergence*, SIAM J. Math. Anal., vol 23, No-6, pp. 1482-1518, 1992.

[3] D.Cioranescu and P.Donota, *An Introduction to Homogenization*, Oxford University Press, 1999.

[4] V.V. Jikov, S. M Kozlov and O.A. Oleinik, *Homogenization of Differential Operators and Integral Functionals*, Springer Verlag, 1994.

[5] L. C. Evans, *Partial Differential Equations*, AMS, 1998.

[6] B.Simon and M.Reed, *Methods of Modern Mathematical Physics. I: Functional Analysis* , Academic Press Inc, 1972.

[7] N.Dirr, *Notes on Measure Theory.*